Jennifer M. Suh

Developing "Algebra-'Rithmetic" in the Elementary Grades

The "Investigations" department features children's hands-on and minds-on explorations in mathematics and presents teachers with open-ended investigations to enhance mathematics instruction. These tasks invoke problem solving and reasoning, require communication skills, and connect various mathematical concepts and principles. The ideas presented here have been tested in classroom settings.

A mathematics investigation—

- has multidimensional content;
- is open ended, with several acceptable solutions;
- is an exploration requiring a full period or longer to complete;
- is centered on a theme or event; and
- is often embedded in a focus or driving question.

In addition, a mathematics investigation involves processes that include—

- researching outside sources;
- collecting data;
- collaborating with peers; and
- using multiple strategies to reach conclusions.

Although this department presents a scripted sequence and set of directions for how this investigation was presented in this particular classroom, *Principles and Standards for School Mathematics* (NCTM 2000) encourages teachers and students to explore multiple approaches and representations when engaging in mathematical activities. This investigation focuses on the importance of algebra in the elementary grades. Students begin an exploration of patterns and functions by reading a Chinese folktale describing a doubling pattern. Next, they identify various functional relationships and, as additional data are revealed, make and test conjectures. Students identify and describe mathematical relationships in a variety of ways and compare and contrast different rates of change. The students use multilink cubes to extend their pattern searching to geometric patterns. This investigation focuses on developing algebraic reasoning by combining concepts from NCTM's Number and Operations standard and its Algebra standard (NCTM 2000) as suggested by *Curriculum Focal Points* (NCTM 2006): "Students [in grade 3] build a foundation for later understanding of functional relationships by describing relationships in context with such statements as, 'The number of legs is 4 times the number of chairs' " (p. 15).

The idea for this series of lessons began with the author's participation in NCTM's E-Workshop "Implementing the Algebra Standard in Grades 3–5." (For information on current NCTM E-Workshop offerings, visit www.nctm.org/profdev.)

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Edited by **Sue McMillen**, *mcmillense@buffalostate.edu, and* **Jodelle S. W. Magner**, *magneris@buffalostate.edu, who teach mathematics and mathematics education courses at Buffalo State College, Buffalo, NY 14222. "Investigations" highlights classroom-tested multilesson units that develop conceptual understanding of mathematics topics. This material can be reproduced by classroom teachers for use with their own students without requesting permission from the National Council of Teachers of Mathematics. Readers are encouraged to submit manuscripts appropriate for this department by accessing* **tcm.msubmit.net**. *Manuscripts must include no more than twelve double-spaced typed pages and two reproducible pages of activities.*

The Investigation

Learning goals, rationale, and pedagogical context

For many people, the word *algebra* conjures up memories of formal symbolic manipulation in their middle or high school mathematics classrooms. However, thinking of algebra only in that context prevents teachers from recognizing the importance of developing algebraic reasoning in early elementary mathematics. *Principles and Standards for*

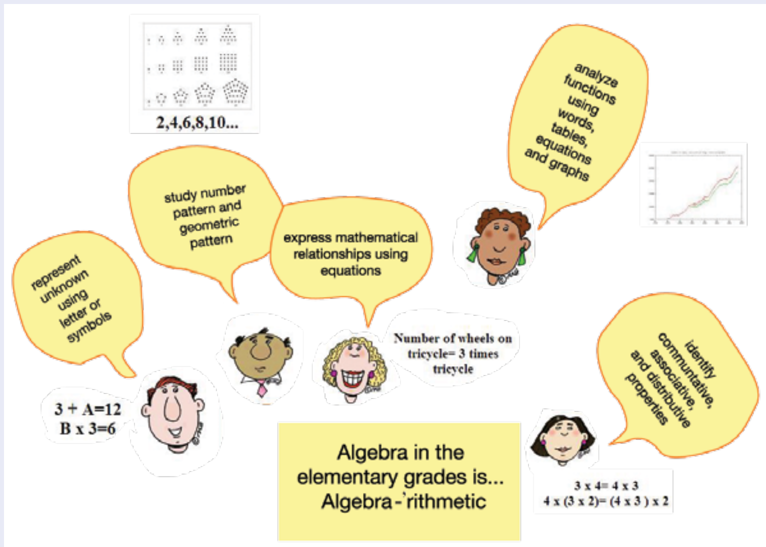
School Mathematics (NCTM 2000) indicates that algebraic reasoning includes the following components: (1) understanding patterns, relations, and functions; (2) representing and analyzing mathematical situations and structures by using algebraic symbols; and (3) analyzing change in various contexts (p. 37). One way to reconceptualize algebra in the elementary grades is to highlight "algebra-'rithmetic"—algebraic reasoning that is embedded in the arithmetic we already teach (see **fig. 1**).

The learning goal for this investigation is for elementary students to develop algebraic reasoning through problem-solving and arithmetic activities. According to Carpenter, Franke, and Levi (2003), the artificial separation of arithmetic and algebra deprives students of powerful ways of thinking about mathematics, especially since the fundamental properties that children use in calculating are the basis for most of the symbolic manipulation in algebra. Integrating algebraic reasoning into arithmetic adds breadth and depth to the existing curriculum. So how does one develop algebraic reasoning? Greenes and Findell (1999) suggest that students develop algebraic reasoning when they are able to interpret algebraic problems by using pictorial, graphic, and verbal description, tables, and numeric representations (see **fig. 2**).

To develop algebraic reasoning and integrate it with other strands of mathematics, these problem-solving activities are designed to heighten students' awareness and ability to search for mathematical patterns and relationships, describe them using multiple representations, and analyze change. The following lessons were taught in a third-grade classroom in Loudoun County, Virginia.

Figure 1

Redefining algebra in the elementary grades



Objectives of the investigation

The students will—

- understand patterns, relations, and functions;
- analyze change in numeric and geometric patterns; and
- represent and analyze mathematical situations and structures by using multiple representations and thus develop algebraic reasoning.

Materials

Lesson 1 (2 days)

For the whole class—

- *Two of Everything* by Lily Toy Hong
- a "magic pot" and other story props

For each student—

- blank cards for input numbers (one per student)

Lesson 2 (1 day)

For each student—

- **activity sheet 1** (see page 253)
- a calculator

Lesson 3 (2 days)

For each student—

- **activity sheet 2** (see page 253)

For each pair of students—

- multilink cubes (about 70 for each pair)
- graph paper
- calculators

Previous knowledge

This investigation builds on opportunities in previous grades in which students learned about repeating and growing patterns through simple numeric and geometric patterns, repetitive songs, chants, and predictive poems. Students had previously identified, described, and applied number patterns and properties in developing strategies for basic facts.

Figure 2

Illustration of the relationships between various types of representations that develop algebraic thinking

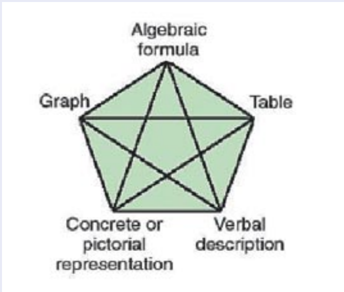


Figure 3

The "magic pot" with story props to reenact the mathematics

Lesson 1: The Magic Pot

Exploring number patterns

To help my students appreciate algebra in their upcoming school year mathematics, I began the first month of school with the theme that mathematics is the search for patterns and relations between numbers. To give the students a context for the following investigation, I read *Two of Everything* by Lily Toy Hong (1993), a retelling of a Chinese folktale. This picture book introduces the mathematical concepts of doubling and of functions. In the story, old farmer Haktak and his wife dig up a large brass pot in their field and discover it has magic doubling powers. However, one day when Mrs. Haktak accidentally falls into the pot, they learn that not everything in life should be doubled. This is a great story to set the stage for exploring number patterns, relationships, and functions. The concept of doubling is critical to teaching addition strategies and is also a prerequisite to understanding multiplication facts for two.

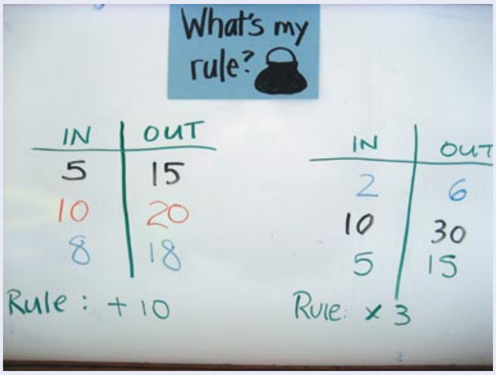
After reading the book, students explored doubling by using a "magic pot" as a conceptual support (see **fig. 3**). The class summarized the story by reenacting the doubling events with a coin purse, a hairpin, and Mr. and Mrs. Haktak. As they retold the story, I had them prepare to keep a record of the items that go into the magic pot. By creating a table showing both input and output, the students found a pattern and generated a rule. At the end of the lesson, we brainstormed about what our class might put in the magic pot in addition to the story items.

The second day, I told the students that the magic pot was doing something different and they had to figure out what it was. I asked them to write a number on a card and drop the card into the magic pot. For example, when a student dropped in a card with the number 5 written on it, I took the card from the back opening of the magic pot and wrote the number 11 on it. The class recorded the numbers on another input-output table. With this list of generated numbers, the students looked for the pattern and the relationship between input and output to determine the function. As the list grew, students made and tested their conjectures. If a few students thought they knew the rule before the rest of the class did, they thought of an input number and silently predicted the output number; they could determine whether they knew the function rule without denying other students an opportunity to discover the numeric pattern.

Depending on the students' level of understanding, you can differentiate the challenge by creating a function rule appropriate for the class (see **fig. 4**). Simple addition and subtraction rules are a good place to start, but more complex rules can be created. Starting the school year with this activity encourages students to constantly search for patterns and relationships as they investigate other concepts throughout the year. Having had this experience, they can move into using more abstract function machine activities from Internet Web sites such as www.mathplayground.com/FunctionMachine.html; nlvm.usu.edu/en/nav/frames_asid_191_g_4_t_2.html; and www.shodor.org/interactivate/activities/WholeNumberCruncher.

Figure 4

What's my rule for the input-output table?



Lesson 2: What Would You Choose?

Analyzing change in number patterns

This lesson uses functions to explore the idea of change in a real-life context. Students experience many situations in their everyday lives that represent constant or varying change. Plant growth and temperature change represent varying change, while the cost of an international call can show constant change. I began the second lesson by asking students if they would rather have 100 coins each day for 10 days or 5 coins and a magic doubling pot that would double only 10 times. Students circled and explained their choices on **activity sheet 1** before they actually worked on the problem. Having them make and test their conjectures gives them ownership of the problem. Initially, many students chose the first option, thinking that 100 coins a day would give them more money. One student said, "I think 100 coins a day is a better deal because 100 is a lot more than 5 doubled."

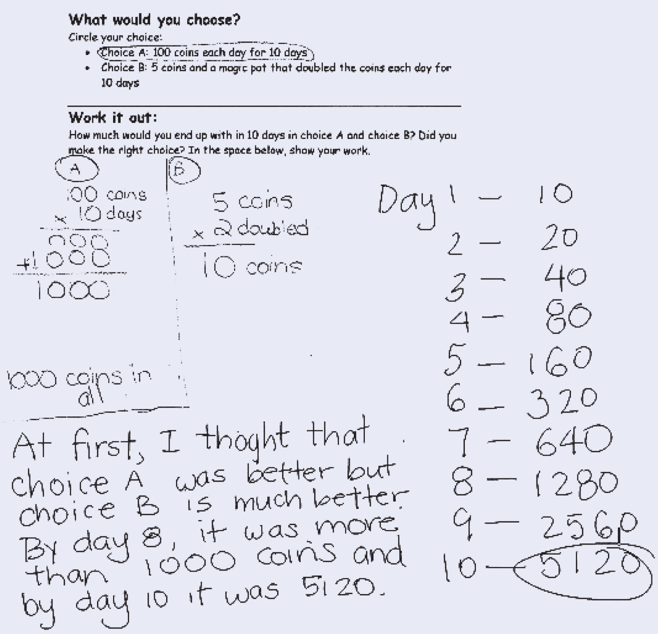
After the students made conjectures and defended their initial reasoning to the class, they explored the problem on their own without my suggesting any strategies. It is important that students grapple with problem solving and try various solution strategies to find the most efficient one. This process also allows students to explore multiple approaches to problem solving. Some students used number sentences, simple charts, or tables to organize their thinking. After the students worked on the problem individually, they talked to their table mates and discussed which approach was a better choice. **Figure 5** shows one student's work on the original version of the activity sheet. Initially, students will verbalize the mathematical relationship informally; this process is the precursor to translating the verbal description into an algebraic formula. Once students become comfortable describing patterns and relationships between numbers, using symbols to represent the variables becomes easier.

To give special needs learners access to this problem, prepare hint cards containing questions to support their learning—for example, "How could you track the money you get each day from choice A?" Organizing information on a table can help students analyze the mathematics in the problem and become efficient problem solvers. Creating a table with the number of days listed from the smallest to the largest number gives students an opportunity to recognize a recursive or iterative pattern.

You can also use technology to illustrate the connection between tables and graphs (see **fig. 6**), allowing for a discussion of how each representation shows the rate of change. Relating this problem to banks or allowances can make this mathematics activity realistic and engaging. To close the lesson, we read *One Grain of Rice* by Demi (1997), which

Figure 5

What would you choose?



What would you choose?

Circle your choice:

- Choice A: 100 coins each day for 10 days
- Choice B: 5 coins and a magic pot that doubled the coins each day for 10 days

Work it out:

How much would you end up with in 10 days in choice A and choice B? Did you make the right choice? In the space below, show your work.

A

100 coins
× 10 days
000
+1 000
1000

1000 coins in all

B

5 coins
× 2 doubled
10 coins

Day 1 — 10
2 — 20
3 — 40
4 — 80
5 — 160
6 — 320
7 — 640
8 — 1280
9 — 2560
10 — 5120

At first, I thought that choice A was better but choice B is much better. By day 8, it was more than 1000 coins and by day 10 it was 5120.

Figure 6

Use of a table and a graph to visually represent rate of change

Days	100 coins a day for ten days	Doubling 5 coins each day with the Magic Pot
1	100	10
2	200	20
3	300	40
4	400	80
5	500	160
6	600	320
7	700	640
8	800	1280
9	900	2560
10	1000	5120

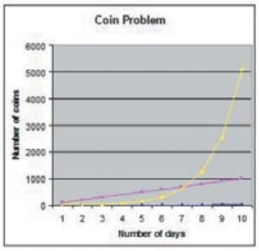
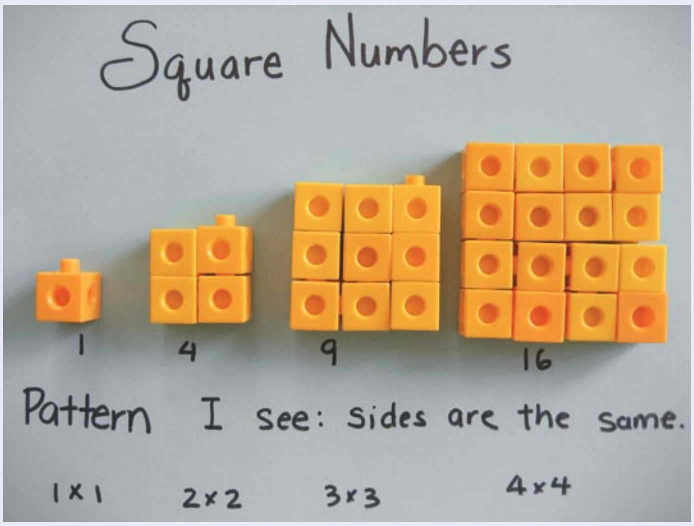


Figure 7

A student's continuation of square number patterns



explores the concept of change and the power of doubling.

Lesson 3: Pattern Seekers

Representing and analyzing geometric patterns

In the third lesson, the students used concrete materials to examine geometric patterns to elaborate on the theme of mathematics as making sense of patterns and relationships of numbers. To introduce the concept of identifying and analyzing geometric patterns, the class examined three squares built from multilink cubes: a 1 × 1, a 2 × 2, and a 3 × 3 square. Students then worked with a partner to create their own set of the squares, continue the pattern (see **fig. 7**), and build the next three squares.

The students created a T-chart on the board as a way to record their data. When asked if they saw a pattern, many gave verbal descriptions such as, "The sides of the squares are the same on all sides." After being prompted to look for a pattern in the number of blocks used for the squares, some commented on the repeated addition rule: "It's like you add the number of blocks: 2 + 2 = 4; 3 + 3 + 3 = 9; 4 + 4 + 4 + 4 = 16." Some gave a multiplication rule; "You multiply the number times the same number, like 1 times 1 equals 1, 2 times 2 equals 4, 3 times 3 equals 9, and 4 times 4 equals 16." Through the students' own verbal descriptions, the teacher was able to write $n \times n$, using the variable n to signify the step number.

The next activity focused on a staircase pattern (see **fig. 8**). The students were provided a variety of materials such as multilink cubes, graph paper, calculators, and a T-chart drawn on the board. Students were asked to think about construction workers building staircases. As the class watched, I built the first three figures in the staircase pattern and then asked pairs of students to build those three plus the next three figures in the pattern. Students talked about the growing pattern as they completed **activity sheet 2**. Asking the students to find at least two ways to describe the pattern encouraged persistence and helped them link various representations.

As the students built each step of the staircase pattern, they were able to see the recursive pattern, as their statements indicated: "Each staircase is growing by the step number" or "Each time I am adding the number of blocks that is the step number to the [number of blocks in the] one before." This recursive relationship was much more apparent to most of the students than the functional relationship. For example, when making a T-chart for the staircase pattern, students readily noticed that they needed to add 6 to the number of blocks in the fifth step to get the number in the sixth step, but they did not necessarily see that they could add the numbers 1 + 2 + 3 + 4 + 5 + 6 to get the number in the sixth step without needing to replicate steps 1 through 5. However, when asked, "Can you predict how many blocks you will need for a 10-step staircase?" some of the students started to make conjectures about their rule and test their thinking without building the tenth staircase. Prompting students to make and test conjectures raised the level of cognitive demand on all students. Despite the fact that most students had to draw the tenth step design and add each step consecutively, a few students were able to figure out the pattern 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 equals 55. Some students even grouped the compatible numbers together, adding 4 + 6 to get 10, 3 + 7 to get 10, 2 + 8 to get 10, and 1 + 9 to get 10 and then adding the leftover 10 and 5 to get 55. The process shows well-designed questions promoting higher-order thinking that can easily differentiate a task to be addressed at multiple levels.

Beyond the Lesson

Many of my third graders enjoyed finding patterns so much that over the next several weeks they worked on an interactive e-module on the National Library of Virtual Manipulatives Web site, nlvm.usu.edu/en/nav/vlibrary.html, to explore other geometric patterns (see **fig. 9**). Although they used this applet, I noticed they were still more comfortable finding a recursive pattern than finding a function rule. Also, although their verbal descriptions of the patterns included many algebraic terms, they needed more guidance to translate their verbal descriptions into an algebraic expression. This ability demonstrated that, given more exposure to similar problems, the students would be quite capable of making that generalization with algebraic symbols. Translating into algebraic expressions is an important mathematics skill as students reason algebraically and work with word problems. Allowing students to rehearse such translations verbally or in writing, from geometric patterns to descriptions of rules and later to simple algebraic equations, can deepen their ability to reason algebraically.

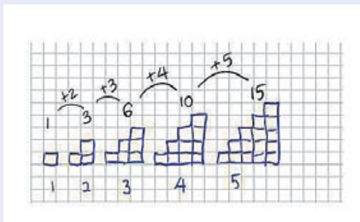
Conclusion

Well-designed "algebra-'rithmetic" investigations in which students express their thinking, make generalizations using words and symbols, and justify their thinking can bring algebraic reasoning to the forefront of many arithmetic activities. Research shows that American students rely heavily on problem-solving strategies such as drawing a picture and guess-and-check, whereas counterparts in other countries that outperform the United States use algebraic reasoning to solve the same problem. Judson (1999) reports that in Japan "teachers [of fifth- and sixth-grade students] show how to arrange data and use mathematical expressions and graphs to help children …[learn] to express the sizes of quantities. Letters such as x and a are introduced" (p. 76). When seeing the variables x and a used in the elementary grades, one might get the impression that students are jumping into abstract concepts they are not ready to understand. But teachers can use investigations such as this as opportunities for

Figure 8

Two students' representations of their solution to the Staircase problem

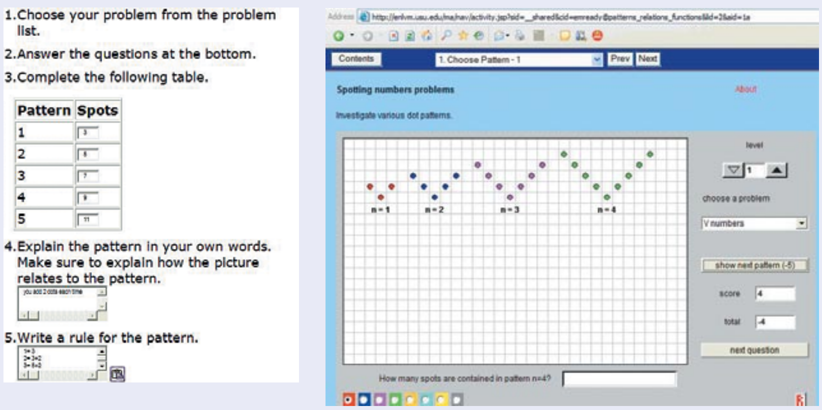
a.



b.

Figure 9

This interactive e-module from the National Library of Virtual Manipulatives allows students to investigate dot patterns.



students to build meaning for and instill confidence in their algebraic reasoning. As Carpenter, Franke, and Levi (2003) state, "Elementary school children are capable of learning how to engage in this type of thinking in mathematics, but often they are not given opportunities to do so" (p. 5).

When students solve problems with manipulative and pictorial models and graphs, they can begin to work with algebraic concepts, as these third graders did by exploring numeric and geometric patterns. They displayed relationships visually, symbolically, numerically, and verbally. Translating among different representations can deepen elementary students' understanding of mathematics. As teachers, we should provide our students with rich algebraic problems and opportunities to represent and solve them in multiple ways that develop their algebraic reasoning abilities.

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Activity Sheet 1. What Would You Choose?

Circle your choice.

- Choice A: 100 coins each day for 10 days
- Choice B: 5 coins and a magic pot that doubled the coins each day for 10 days

Work it out: How many coins would you have in 10 days with choice A? With choice B? Show your work on the back of this page.

Final thought: After working through the problem, would you still make the same choice? Why or why not? _____

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From the November 2007 issue of TEACHING CHILDREN Mathematics

Activity Sheet 2. Pattern Seekers

1. On the graph paper below, draw the pattern you see.

2. Create a table of the pattern you see.

Step	1	2	3	4	5	6
Number of Cubes						

3. On the back of this page, describe the pattern in your own words. Then write a rule for the pattern.

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