Reality Check 11 Problem 3

We need to prove that assume that Z_1 and Z_2 are each multiplied componentwise by the entire windowing function h, and NMZ_1 and NMZ_2 in equation (11.38) are each multiplied componentwise by h, that equation (11.39) still holds.

Let $Z_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $Z_2 = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$. Let $h_j = \sqrt{2} sin \frac{(j-\frac{1}{2})\pi}{2n}$ for $j = 1, \ldots, 2n$.

We multiply Z_1 and Z_2 componentwise by the entire windowing function h. Then we have $Z_1 = \begin{bmatrix} h_1x_1 \\ h_2x_2 \\ h_3x_3 \\ h_4x_4 \end{bmatrix}$ and $Z_2 = \begin{bmatrix} h_1x_3 \\ h_2x_4 \\ h_3x_5 \\ h_4x_6 \end{bmatrix}$.

Then $NMZ_1 = \begin{bmatrix} h_1x_1 - Rh_2x_2 \\ -Rh_1x_1 + h_2x_2 \\ h_3x_3 + Rh_4x_4 \\ Rh_3x_3 + h_4x_4 \end{bmatrix}$ and $NMZ_2 = \begin{bmatrix} h_1x_3 - Rh_2x_4 \\ -Rh_1x_3 + h_2x_4 \\ h_3x_5 + Rh_4x_6 \\ Rh_3x_5 + h_4x_6 \end{bmatrix}$.

We multiply NMZ_1 and NMZ_2 componentwise by the entire windowing function h. Then we have $NMZ_1 = \begin{bmatrix} h_1^2x_1 - h_1Rh_2x_2 \\ -h_2Rh_1x_1 + h_2^2x_2 \\ h_3^2x_3 + h_3Rh_4x_4 \\ h_4Rh_3x_3 + h_4^2x_4 \end{bmatrix}$ and $NMZ_2 = \begin{bmatrix} h_1^2x_3 - h_1Rh_2x_4 \\ -h_2Rh_1x_3 + h_2^2x_4 \\ h_3^2x_5 + h_3Rh_4x_6 \\ h_4Rh_3x_5 + h_4^2x_6 \end{bmatrix}$.

We take

$$\frac{1}{2}(NMZ_1)_2 + \frac{1}{2}(NMZ_2)_0 = \frac{1}{2}(h_3^2 x_3 + h_3 Rh_4 x_4) + \frac{1}{2}(h_1^2 x_3 - h_1 Rh_2 x_4) \tag{1}$$

Observe that $Rh_i x_n = h_{2n-i+1} x'_n$, where x'_n is the inverse of x_n.

Then

$$\begin{aligned} \frac{1}{2}(h_3^2 x_3 + h_3 Rh_4 x_4) + \frac{1}{2}(h_1^2 x_3 - h_1 Rh_2 x_4) &= \frac{1}{2}(h_3^2 x_3 + h_3 h_1 x'_4) + \frac{1}{2}(h_1^2 x_3 - h_1 h_3 x'_4) \\ &= \frac{1}{2} x_3 (h_3^2 + h_1^2) \end{aligned} \tag{2}$$

Observe that

$$\begin{aligned} h_3^2 + h_1^2 &= (\sqrt{2})^2((sin\frac{(1-\frac{1}{2})\pi}{2n})^2 + (sin\frac{(3-\frac{1}{2})\pi}{2n})^2) \\ &= 2((sin\frac{(1-\frac{1}{2})\pi}{4})^2 + (sin\frac{(1-\frac{1}{2})\pi}{4} - \frac{\pi}{2})^2) \\ &= 2((sin\frac{(1-\frac{1}{2})\pi}{4})^2 + (cos\frac{(1-\frac{1}{2})\pi}{4})^2) \\ &= 2 * (1) \\ &= 2. \end{aligned} \tag{3}$$

Therefore, from equation (1), (2) and (3),

$$\frac{1}{2}(NMZ_1)_2 + \frac{1}{2}(NMZ_2)_0 = \frac{1}{2}(2 * x_3) = x_3$$

.

Apply the same logic for $\frac{1}{2}(NMZ_1)_3 + \frac{1}{2}(NMZ_2)_1$, we could obtain

$$\frac{1}{2}(NMZ_1)_3 + \frac{1}{2}(NMZ_2)_1 = x_4.$$

Thus, $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \frac{1}{2}(NMZ_1)_{n,\ldots,2n-1} + \frac{1}{2}(NMZ_2)_{0,\ldots,n-1}$, equation (11.39) still holds.