Computational and Statistical Methods in Finance

An Introduction and Overview Tutorial

James E. Gentle
George Mason University
Contact: jgentle@gmu.edu
Why Study Financial Data?

• To get rich
  (just kidding!)

• Real and interesting challenges of statistical modeling
  (most accepted models begin with the premise that one
  cannot get rich by using the analyses)

• To maintain fair and orderly markets
  (government regulators and investors, especially people
  approaching or in retirement, are particularly interested in
  this aspect)
Financial Data and Development of Statistical Methodology

Many early applications of statistics involved financial data.

Many advances in time series analysis have been motivated by study of financial data.

Methods for statistical graphics were developed to aid in the analysis of financial data.

Interesting approaches to data mining have been developed and applied to financial data long before the term “data mining” was used.
The Appeal of Financial Data

Whether a statistician has any interest in financial applications or not, financial data have many fascinating properties that are of interest to statisticians.

Providing further appeal to the statistician or data analyst is the ready availability of interesting data.
Outline

• What is “computational finance”?

• Financial data: its nature and sources

• Statistical models of asset prices

• Pricing derivative assets

• The nature of volatility, and beyond Black-Scholes

• Interesting areas of research in computational finance

The presentation is very basic, and assumes no prior knowledge of Finance.
What is Computational Finance?

“Finance” includes accounting, corporate structure, market structures, investments and pricing of assets, government finance, ..., and the list goes on.

“Computational Finance”, which is also sometimes called “Financial Engineering”, is distinguished by the methodology.

It makes heavy use of statistical modeling and Monte Carlo techniques.

Statistical learning in financial applications is also part of computational finance.

These methods are not very useful in accounting, corporate structure, and so on.

Computational finance is primarily concerned with pricing of assets and other topics in investments.
Financial Data
Sources of Financial Data

Proprietary and expensive — or free and readily available.

Some financial data are difficult to obtain; some have been collected at great expense, and may be expected to provide some monetary advantage to their possessors.

For general academic interests there is a wealth of easily available data.

The prices at which securities have traded, at least on a daily basis, are easily available.

These securities include the basic shares of stock as well as various derivatives of these shares.

One of the easiest places to get current or historical stock price data is
http://finance.yahoo.com/
Sources and Diversity of Financial Data

There are many different types of financial data including balance sheets, earnings statements and their various details, stock prices, and so on.

It also includes names and backgrounds of directors and other company officers, news items relating to the company, general economic news, and so on.

Government agencies often require that companies whose stock is publicly traded file periodic reports of particular financial characteristics of those companies.

Usually available at the agencies’ websites.
Diversity of Financial Data

Two major components of financial data are the opinions of financial analysts and the chatter of the large army of people with a computer and a connection to the internet where they can post touts.

This type of data must be included in the broad class of financial data.

They actually have an effect on other financial data such as stock prices.

They present very interesting problems for data mining.
Types of Financial Data and Data Quality

Three of the general types of financial data are numerical. One type of numerical data are prices and volume in trading of financial assets.

This kind of data is objective and highly reliable.

Two other types of numerical data have to do with the general financial state of an individual company and of the national and global economy.

These kinds of data, while ostensibly objective, depend on the subjectivity of the definitions of the terms.

A fourth type of financial data is text data about company officers, products or services of the company, and so on.

Finally, another type of text data are the statements and predictions by anyone with the ability to publicize anything.
How to Handle Data of Varying Quality

When it is possible to assign relative variances to different subsets of numerical data, the data can easily be combined using weights inversely proportional to the variance.

The differences in the nature of financial data, however, make it difficult to integrate the data in any meaningful way.

The totality of relevant financial data is difficult to define.
Data of Varying Quality

The provenance of much financial data is almost impossible to track.

Rumors easily become data.

Even “hard data”, that is, numerical data on book values, earnings, and so on, are not always reliable.

A PE ratio for example may be based on “actual earnings” in some trailing period or “estimated earnings” in some period that includes some future time. Unfortunately, “earnings”, even “actual”, is a rather subjective quantity, subject to methods of booking.

(Even if the fundamental operations of the market were efficient, as is assumed in most academic models, the basic premise of efficiency, that is, that everyone has the same information, is not satisfied. Analysts and traders with the ability to obtain really good data have an advantage.)
Seeking the Unexpected in DisparateDatasets

What we often now call data mining has a long history in financial applications.

In 1979, someone discovered that the US stock market (as measured by the Dow Industrials) rose during years in which the Super Bowl (an annual playoff of two American football teams begun in 1967) was won by teams of one type and fell when it was one by teams of the other type.

(Specifically, it rose if a team from the original NFL won, and it fell if a team from the AFL won.)

The Super Bowl Stock Market Predictor has been amazingly accurate (except for a bad run of 4 years in 1998–2001).
An interesting discovery that resulted from mining of financial data is called the “January effect”.

Several years ago, it was discovered that there are anomalies in security prices during the first few days of January.

There are various details in the differences of January stock prices and the prices in other months.

The most relevant general fact is that for a period of over 80 years the average rate of return of the major indexes of stock prices during January was more than double the average rate of return for any other month.
The Uncertainty Principle in Financial Markets

While the discovery of the January effect came from just an obvious and straightforward statistical computation, it serves to illustrate a characteristic of financial data.

I call it the “uncertainty principle”, in analogy to Heisenberg’s uncertainty principle that states that making a measurement affects the state of the thing being measured.

After the January effect became common knowledge, the effect seemed to occur earlier (the “Santa Claus rally”).

This, of course, is exactly what one might expect. Carrying such expectations to the second order, the Santa Claus rally may occur in November, and then perhaps in October.
Properties of Financial Data

A basic type of financial data is the price of an asset, and a basic property of the price is that it fluctuates.

Anything that is openly traded has a market price that may be more or less than some “fair” price.

In financial studies, a general objective is to measure “fair price”, “value”, or “worth”.

For shares of stock, the fair price is likely to be some complicated function of intrinsic (or “book”) current value of identifiable assets owned by the company, expected rate of growth, future dividends, and other factors.
There is a certain amount of jargon used for other aspects of the price.

Both “risk” and “volatility” refer to the variation in price. Either the standard deviation or the variance of the price may be called “risk”. (I am not being precise here; “variance” and “standard deviation” of course have very precise meanings in probability theory, but my use of the term must be interpreted more loosely. “Risk”, especially when defined as a variance, may be decomposed into interesting components, such as market risk and the risk of a particular asset over and above the market risk.)

The term “stochastic volatility” refers to a situation in which the volatility (“standard deviation”) is changing randomly in time.
Stylized Properties of Financial Data

Some of the characteristics of financial data that make it interesting and challenging to analyze are the following.

- Heavy tails. The frequency distribution of rates of return decrease more slowly than $\exp(-x^2)$.

- Asymmetry in rates of return. Rates of return are slightly negatively skewed. (Because traders react more strongly to negative information than to positive information.)

- Asymmetry in lagged correlations. Coarse volatility predicts fine volatility better than the other way around.
Stylized Properties of Financial Data (Continued)

• Clustering of volatility.

• Aggregational normality.

• Quasi long range dependence.

• Seasonality.
Variation in Asset Prices

The price, either the market price or the fair price, varies over time. We often assume discrete time, \( t_0, t_1, t_2, \ldots \) or \( t, t + 1, t + 2, \ldots \).

The prices of individual securities, even if they follow similar models, behave in a way peculiar to the security.

The prices are generally rather strongly positively correlated, however.
Indexes of Asset Prices

We often study the prices of some index of stock prices.

There are more security-specific extraordinary events that affect the price of a given security, than there are extraordinary events that affect the overall market.

The S&P 500 is a commonly-used index.

The PSI-20 (Portuguese Stock Index) is an index of 20 companies that trade on Euronext Lisbon.
Modeling Prices
Models of Asset Prices

A stochastic model of the price of a stock may view the price as a random variable that depends on previous prices and some characteristic parameters of the particular stock.

For example, in discrete time:

$$S_{t+1} = f(S_t, \mu, \sigma) \tag{1}$$

where $t$ indexes time, $\mu$ and $\sigma$ are parameters, and $f$ is some function that contains a random component.

The randomness in $f$ may be assumed to reflect all variation in the price that is not accounted for in the model.

There are two basic types of models of asset prices.

One type is a stochastic diffusion differential equation and the other is an autoregressive model.
Models of Relative Changes of Stock Prices

In the absence of exogenous forces, the movement of stock prices is usually assumed to be some kind of random walk.

A simple random walk would have step sizes independent of location and furthermore could take the prices negative.

Also, it seems intuitive that the random walk should have a mean step size that is proportional to the magnitude of the price.

The proportional rate of change, that is, the rate of return,

\[ \frac{(S_{t+1} - S_t)}{S_{t+1}} \]

is more interesting than the prices themselves, and is more amenable to fitting to a probability model.
July 1987 to June 2008

SPX daily returns

-0.20 -0.15 -0.10 -0.05 0.00 0.05 0.10


July 1987 to June 2008
Diffusion Models

A good general model of a random walk is a Brownian motion, or a Wiener process; hence, we may write the model as a drift and diffusion,

$$\frac{dS(t)}{S(t)} = \mu(S(t), t)dt + \sigma(S(t), t)dB,$$

where $dB$ is a Brownian motion, which has a standard deviation of 1. The standard deviation of the rate of change is therefore $\sigma(S(t), t)$.

In this equation both the mean drift and the standard deviation may depend on both the magnitude of the price $S(t)$ and also on the time $t$ itself.
Simple Diffusion Model

We often assume that $\mu(\cdot)$ and $\sigma(\cdot)$ do not depend on the value of the state and that they are constant in time, we have the model

$$
\frac{dS(t)}{S(t)} = \mu dt + \sigma dB.
$$

A “geometric Brownian motion”.

In this form,
$\mu$ is called the drift and the diffusion component and
$\sigma$ is called the volatility.
Geometric Brownian Motion

We note that as a model for the rate of return, $dS(t)/S(t)$ geometric Brownian motion is similar to other common statistical models:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t)$$

or

$$\text{response} = \text{systematic component} + \text{random error}.$$ 

Also, note that without the stochastic component, the differential equation has the simple solution

$$S(t) = ce^{\mu t},$$

from which we get the formula for continuous compounding for a rate $\mu$. 
More on the Geometric Brownian Motion Model

The rate of growth we expect for $S$ just from the systematic component in the geometric Brownian motion model is $\mu$.

Because the expected value of the random component is 0, we might think that the overall expected rate of growth is just $\mu$.

Closer analysis, in which we consider the rate of change $\sigma$ being equally likely to be positive or negative and the effect on a given quantity if there is an uptick of $\sigma$ followed by a downtick of equal magnitude yields a net result of $-\sigma^2$ for the two periods.

The average over the two periods therefore is $-\sigma^2/2$. The stochastic component reduces the expected rate of $\mu$ by $-\sigma^2/2$.

This is the price of risk.
Estimation of Model Parameters

We can estimate $\mu$ and $\sigma$ from the historical rates of return; the mean and the standard deviation respectively.

Although it is easy to understand the model, it is not so obvious how we would estimate the parameters.

First of all, obviously, we cannot work with $dt$. We must use finite differences based on $\Delta t$, but the question is how long is $\Delta t$?

An interesting property of Brownian motion is that its variation seems to depend on the frequency at which it is observed; it is infinitely “wiggly”. (Technically, the first variation of Brownian motion is infinite.)
Autoregressive Models of Relative Changes of Stock Prices

The most widely-used time series models in the time domain are those that incorporate either a moving average or an autoregression term.

These ARMA models are not even close approximations to observational data.

The main problem seems to be that the assumption of a constant variance does correspond to reality.

Variations, such as ARCH or GARCH models, models seem to fit the data better.
Fitting the Models

Returning now to the problem of estimating a parameter of a continuous-time process, we consider the similar model for continuous compounding (deterministic growth).

If an amount $A$ is invested for $n$ periods at a per-period rate $R$ that is compounded $m$ times per period, the terminal value is

$$A \left(1 + \frac{R}{m}\right)^{nm}.$$  

The limit of the terminal value as $m \to \infty$ is

$$Ae^{Rn}.$$  

This suggests that $\Delta t$ can be chosen arbitrarily, and, since rates are usually quoted on an annualized basis, we chose $\Delta t$ to be one year.
Fitting the Models

Using the formula for compounded interest, we first transform the closing price data $S_0, S_1, S_2 \ldots$ to $r_i = \log(S_i/S_{i-1})$.

An obvious estimator of the annualized volatility, $\sigma$, based on $N$ periods each of length $\Delta t$ (measured in years) is

$$\tilde{\sigma} = \frac{1}{\sqrt{\Delta t}} \sqrt{\frac{1}{N} \sum_{i=1}^{N} (r_i - \bar{r})^2}.$$  

We are now in a position to use the geometric Brownian motion drift-diffusion model (with the simplifying assumptions of constant drift and diffusion parameters).
Solution of the Stochastic Differential Equation

The solution of a differential equation is obtained by integrating both sides and allowing for constant terms.

Constant terms are evaluated by satisfying known boundary conditions, or initial values.

In a stochastic differential equation (SDE), we must be careful in how the integration is performed, although different interpretations may be equally appropriate.
The SDE

The SDE defines a simple Ito process with constant coefficients,

\[ dS(t) = \mu S(t)dt + \sigma S(t)dB(t). \]

We solve this using Ito’s formula (which is relatively standard fare in the stochastic calculus).

Integrating to time \( t = T \), we have

\[ S(T) = S(t_0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma \Delta B \right). \]

Numerical methods for the solution of SDEs are not as well-developed as those for deterministic differential equations.
Derivatives
Modeling Prices of Derivative Assets: Insurance and Futures

For any asset whose price varies over time, there may be a desire to insure against loss, or there may be interest in putting up a small amount of capital now so as to share in future gains.

The insurance or the stake that depends on an “underlying asset” is itself an asset that can be traded.

There are many ways the insurance or stake can be structured.

It most certainly would have an expiration date.

It could be a contract or it could be an option.

It could be exercisable only at a fixed date or at anytime before expiration.
Forward Contracts

One of the simplest kinds of derivative is a forward contract.

This is an agreement to buy or sell an asset at a specified time at a specified price.

The agreement to buy is a long position, and the agreement to sell is a short position.

The agreed upon price is the delivery price.

The consummation of the agreement is an execution.

Forward contracts are relatively simple to price, and their analysis helps in developing pricing methods for other derivatives.
Forward Contracts

Let $k$ be the delivery price or “strike price” at the settlement time $t_s$, and let $X_t$ be the value of the underlying. (We consider all times to be measured in years.)

Payoff of a Forward Contract
Pricing Forward Contracts

Consider a forward contract with parameters $k$, $t_s$, and $X_t$.

What is the fair price of this contract today, at time $t$?
A basic assumption in pricing financial assets is that there exists a fixed rate of return on some available asset that is constant and “guaranteed”; that is, there exists an asset that can be purchased, and the value of that asset changes at a fixed, or “riskless”, rate.

The concept of “risk-free rate of return” is a financial abstraction based on the assumption that there is some absolute controller of funds that will pay a fixed rate of interest over an indefinite length of time.

Well, let’s consider a spherical cow —
Okay, let’s just pretend …

In current world markets, the rate of interest on a certain financial instrument issued by the United States Treasury is used as this value.

Given a riskless positive rate of return, another key assumption in economic analyses is that there are no opportunities for arbitrage.

An arbitrage is a trading strategy with a guaranteed rate of return that exceeds the riskless rate of return.

In financial analysis, we assume that arbitrages do not exist. This is the “no-arbitrage principle”.
The No-Arbitrage Principle

This follows from an assumption that the market is “efficient”.

(If you believe that, have I got a deal for you! See me after this session.)

The spherical cow again —
More Pretensions and Where They Lead

Let’s also assume that all markets have two liquid sides and that there are market participants who can establish positions on either side of the market.

The assumption of a riskless rate of return leads to the development of a replicating portfolio, having fluctuations in total valuation that can match the expected rate of return of any asset.

The replication approach to pricing derivatives is to determine a portfolio and an associated trading strategy that will provide a payout that is identical to that of the underlying.

This portfolio and trading strategy replicates the derivative.
Finally — Pricing Forward Contracts

Consider a forward contract that obligates one to pay $k$ at $t_s$ for the underlying.

The value of the contract at expiry is $X_{t_s} - k$, but of course we do not know $X_{t_s}$.

If we have a riskless (annual) rate of return $r$, we can use the no-arbitrage principle to determine the correct price of the contract.

To apply the no-arbitrage principle, consider the following strategy:

- take a long position in the forward contract;
- take a short position of the same amount in the underlying (sell the underlying short).
Implementing the Strategy

With this strategy, the investor immediately receives $X_t$ for the short sale of the underlying.

At the settlement time $t_s$, this amount can be guaranteed to be

$$X_t e^{r(t_s - t)}$$

using the risk-free rate of return.
The No-Arbitrage Principle and the Fair Price

Now, if the settlement price \( k \) is such that

\[
k < X_t e^{r(t_s - t)},
\]

a long position in the forward contract and a short position in the underlying is an arbitrage, so by the no-arbitrage principle, this is not possible.

Conversely, if

\[
k > X_t e^{r(t_s - t)},
\]

a short position in the forward contract and a long position in the underlying is an arbitrage, and again, by the no-arbitrage principle, this is not possible.

Therefore, under the no-arbitrage assumption, the correct value of the forward contract, or its “fair price”, at time \( t \) is \( X_t e^{r(t_s - t)} \).
Other Derivatives

There are several modifications to the basic forward contract that involve different types of underlying, differences in when the agreements can be executed, and differences in the nature of the agreement: whether it conveys a right (that is, a “contingent claim”) or an obligation.

The common types of derivatives include stock options, index options, commodity futures, and rate futures.

Stock options are used by individual investors and by investment companies for leverage, hedging, and income.

Index options are used by individual investors and by investment companies for hedging and speculative income.

Commodity futures are used by individual investors for speculative income, by investment companies for income, and by producers and traders for hedging.
Puts and Calls

Common derivatives are “puts” and “calls” on stocks or stock indexes.

A put is the right to sell; a call is the right to buy.

They are “options”.

There are various ways that an option agreement can be structured, and so there are different types of options.

“European” style options can only be exercised on the expiration date.

“American” style options carry the right to exercise anytime between the time of acquisition of the right and the expiration date.
Pricing Puts and Calls

How to price a derivative is a difficult question.

A model for the “value” of an option may be expressed as an equation in the form

\[ V(t) = g(S(t), \mu(t), \sigma(t)), \]

where \( V(t) \) is “value”, that is, “correct price” of the option at time \( t \) (relative to the expiration date), \( S(t) \) is market price of the underlying at time \( t \), \( \mu(t) \) and \( \sigma(t) \) are parametric characteristics of the underlying, and \( g \) is some function that contains a random component.

As usual, we assume frictionless trading; that is, we ignore transaction costs.
Pricing Puts and Calls

\[ V(t) = g(S(t), \mu(t), \sigma(t)). \]

The randomness in \( g \) may be assumed to reflect all variation in the price that is not accounted for in the model.

The price of the underlying will fluctuate, and so the price of the derivative is related to the expected value of the underlying at expiration.
The Price of a European Call Option

A European call option is a contract that gives the owner the right to buy a specified amount of an underlying for a fixed strike price, \( K \) on the expiration or maturity date \( T \).

The owner of the option does not have any obligations in the contract.

The payoff, \( h \), of the option at time \( T \) is either 0 or the excess of the price of the underlying \( S(T) \) over the strike price \( K \).

Once the parameters \( K \) and \( T \) are set, the payoff is a function of \( S(T) \):

\[
h(S(T)) = \begin{cases} 
  S(T) - K & \text{if } S(T) > K \\
  0 & \text{otherwise}
\end{cases}
\]

The price of the option at any time is a function of the time \( t \), and the price of the underlying \( s \). We denote it as \( P(t, s) \).
The Price of a European Call Option

We wish to determine the fair price at time $t = 0$.

It seems natural that the price of the European call option should be the expected value of the payoff of the option at expiration, discounted back to $t = 0$:

$$P(0, s) = e^{-rT} E(h(S(T))).$$

The basic approach in the Black-Scholes pricing scheme is to seek a portfolio, with zero expected value, that consists of short and/or long positions in the option, the underlying, and a risk-free bond.
Pricing Puts and Calls

In developing pricing formulas for derivatives, we adopt the same two key principles as before:

• no-arbitrage principle

• replicating, or hedging, portfolio
Let’s Pretend

One does not need any deep knowledge of the market to see that this assumption does not hold, but without the assumption it would not be possible to develop a general model.

Every model would have to provide for input from different levels of information for different participants; hence, the model would necessarily apply to a given set of participants.

While the hypothesis of an efficient market clearly cannot hold, we can develop useful models under that assumption.

(The situation is as described in the quote often attributed to George Box: “All models are wrong, but some are useful.”)
Okay. Let’s Proceed

There are two essentially equivalent approaches to determining the fair price of a derivative, use of delta hedging and use of a replicating portfolio.

Let’s briefly consider replicating portfolios, as before.

The replication approach is to determine a portfolio and an associated trading strategy that will provide a payout that is identical to that of the underlying.

This portfolio and trading strategy replicates the derivative.
Replicating Strategies

A replicating strategy involves both long and short positions.

If every derivative can be replicated by positions in the underlying (and cash), the economy or market is said to be complete.

We will generally assume complete markets.

The Black-Scholes approach leads to the idea of a self-financing replicating hedging strategy.

The approach yields the interesting fact that the price of the call does not depend on the expected value of the underlying.

It does depend on its volatility, however.
**Expected Rate of Return on Stock**

Assume that XYZ is selling at $S(t_0)$ and pays no dividends.

Its expected value at time $T > t_0$ is merely the forward price for what it could be bought now, where the forward price is calculated as

$$e^{r(T-t_0)}S(t_0),$$

where $r$ is the risk-free rate of return, $S(t_0)$ is the spot price, and $T - t_0$ is the time interval.

This is an application of the no-arbitrage principle.

The holder of the forward contract (long position) on XYZ must buy stock at time $T$ for $e^{r(T-t_0)}S(t_0)$, and the holder of a call option buys stock only if $S(T) > K$. 
Expected Rate of Return on Stock

Now we must consider the role of the volatility.

For a holder of forward contract, volatility is not good, but for a call option holder volatility is good, that is, it enhances the value of the option.

Under the assumptions above, the volatility of the underlying affects the value of an option, but the expected rate of return of the underlying does not.
Let’s Look at This Another Way

A simple model of the market assumes two assets:

- a *riskless asset* with price at time $t$ of $\beta_t$

- a *risky asset* with price at time $t$ of $S(t)$.

The price of a derivative can be determined based on trading strategies involving these two assets.
A Replicating Portfolio of the Two Asset Classes

The price of the riskless asset follows the deterministic ordinary differential equation

\[ d\beta_t = r\beta_t dt, \]

where \( r \) is the instantaneous riskfree interest rate.

The price of the risky asset follows the stochastic differential equation

\[ dS(t) = \mu S(t)dt + \sigma S(t)dB_t. \]

We have a portfolio \( p = (p_1, p_2) \):

\[ p_1\beta_t + p_2 S(t). \]
Okay. So What’s the Value of a Call?

We have then for the value (fair price) of the call to be

\[ C(t) = \Delta_t S(t) - e^{r(T-t)} R(t), \]

where \( R(t) \) is the current value of a riskless bond.
The Portfolio

A portfolio is a vector $p$ whose elements sum to 1.

By the no-arbitrage principle, there does not exist a $p$ such that for some $t > 0$, either

- $p^T s_0 < 0$ and $p^T S(t)(\omega) \geq 0$ for all $\omega$,

or

- $p^T s_0 \leq 0$ and $p^T S(t)(\omega) \geq 0$ for all $\omega$, and $p^T S(t)(\omega) > 0$ for some $\omega$. 
Can We Actually Apply a Replicating Strategy?

A derivative $D$ is said to be *attainable* (over a universe of assets $S = (S^{(1)}, S^{(2)}, \ldots, S^{(k)})$) if there exists a portfolio $p$ such that for all $\omega$ and $t$,

$$D_t(\omega) = p^T S(t)(\omega).$$

Not all derivatives are attainable.

The replicating portfolio approach to pricing derivatives applies only to those that are attainable.

So, ... let’s pretend.
Other Issues for a Replicating Strategy

The value of a derivative changes in time and as a function of the value of the underlying; therefore, a replicating portfolio must be changing in time or “dynamic”.

(Note that transaction costs are ignored!)

The replicating portfolio is self-financing; that is, once the portfolio is initiated, no further capital is required. Every purchase is financed by a sale.

If the portfolio is self-financing

\[ d(a_t S(t) + b_t e^{rt}) = a_t dS(t) + r b_t e^{rt} dt. \]
The Black-Scholes Differential Equation

Here’s the idea that changed the change the way many traders went about their business 30 years ago.

Consider the fair value \( V \) of a European call option at time \( t < T \).

At any time this is a function of both \( t \) and the price of the underlying \( S_t \). We would like to construct a dynamic, self-financing portfolio \((a_t, b_t)\) that will replicate the derivative at maturity.

If we can, then the no-arbitrage principle requires that

\[
a_t S_t + b_t e^{rt} = V(t, S_t),
\]

for \( t < T \).

We assume no-arbitrage and we assume that a risk-free return is available.
... continued ...

Assuming $V(t, S_t)$ is continuously twice-differentiable, we differentiate both sides of the equation that represents a replicating portfolio with no arbitrage:

$$a_t dS_t + rb_te^{rt}dt = \left( \frac{\partial V}{\partial S_t} \mu S_t + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S^2_t \right) dt$$

$$+ \frac{\partial V}{\partial S_t} (\sigma S_t) dB_t$$

By the market model for $dS_t$ the left-hand side is

$$(a_t \mu S_t + rb_t e^{rt})dt + a_t \sigma S_t dB_t.$$ 

Equating the coefficients of $dB_t$, we have

$$a_t = \frac{\partial V}{\partial S_t}.$$ 

From our equation for the replicating portfolio we have

$$b_t = (V(t, S_t) - a_t S_t) e^{-rt}.$$
Now, equating coefficients of $dt$ and substituting for $a_t$ and $b_t$, we have the Black-Scholes differential equation for European calls,

$$r \left( V - S_t \frac{\partial V}{\partial S_t} \right) = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}$$

Notice that $\mu$ is not in the equation!

(Similar for puts.)
Solving the Black-Scholes SDEs

The solution of the differential equation depends on the boundary conditions.

In the case of European options, these are simple.

For calls, they are

$$V_c(T, S_t) = (S_t - K)^+,$$

and for puts, they are

$$V_p(T, S_t) = (K - S_t)^+.$$

With these boundary conditions, there are closed form solutions to the Black-Scholes differential equation.
The Black-Scholes Formula

For the call, for example, it is

$$C_{BS}(t, S_t) = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2),$$

where

$$d_1 = \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}},$$

$$d_2 = d_1 - \sigma \sqrt{T - t},$$

and

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-y^2/2} dy.$$ 

$C_{BS}$ is the correct, or fair, price of the call option at time $t$.

That was easy!
How Did We Get the Black-Scholes Formula?

Now back to reality.

We recall the assumptions of the Black-Scholes model:

- differentiability of stock prices with respect to time

- a dynamic replicating portfolio can be maintained without transaction costs

- returns are
  - independent
  - normal
  - mean stationary
  - variance stationary
Beyond Black-Scholes
Assessment of Models for Price Movements

Do observed rates of return behave this way?

Looking back at our plot, the data do not appear to meet the assumptions of the model.

We can notice violations of the assumptions:

• the data have several outliers

• the data are asymmetric

• the data values seem to cluster

• the data are not normal (Gaussian)
Normal Q–Q Plot

Sample Quantiles

Theoretical Quantiles
Assessment of Models for Price Movements

Although the data seem to be centered about 0, and there seem to be roughly as many positive values as negative values, there appear to be larger extreme negative values than extreme positive values.

The extreme values are more than what we would expect in a random sample from a normal distribution.

The crash of October 19, 1987, is an extreme outlier; but there are others. Finally, the data do not seem to be independent; specifically, the extreme values seem to be clustered.

The large drop on October 19, 1987, is followed by a large gain.

All of these empirical facts seem to bring into question the whole idea that the market is efficient.

Either basic facts are changing rapidly, or there are irrational motives at play.
Assessment of Models for Price Movements

The returns in this example are for an index of 500 large stocks.

We would expect this index to behave more in line with any model based on assumptions of independent normality than the price of some individual stock.

These kinds of violations of the assumptions, however, can be observed in other indexes as well as in individual stock prices.

Both standard stochastic differential equations and simple autoregressive processes have a constant variance, but such models do not correspond well with empirical observations.

This means that neither a single SDE nor a stationary autoregressive model will be adequate.
Another Approach

And a Way to Assess the Failures of Assumptions
Monte Carlo Methods in Finance

In Monte Carlo methods, we simulate observations from the model, and then perform operations on those pseudo-observations in order to make inferences on the population being studied.

Monte Carlo methods require a source of random numbers and methods to transform a basic sequence of random numbers to a sequence that simulates any particular distribution.
Monte Carlo Estimation

In its simplest form, Monte Carlo simulation is the evaluation of a definite integral

\[
\theta = \int_D f(x) \, dx
\]

by identifying a random variable \( Y \) with support on \( D \) and density \( p(y) \) and a function \( g \) such that the expected value of \( g(Y) \) is \( \theta \):

\[
E(g(Y)) = \int_D g(y)p(y) \, dy = \int_D f(y) \, dy = \theta.
\]

The problem of evaluating the integral becomes the familiar statistical problem of estimating a mean, \( E(f(Y)) \).
Monte Carlo Estimation

The statistician quite naturally takes a random sample and uses the sample mean.

In a simple univariate case where $D$ is the interval $(a, b)$, for a sample of size $m$, an estimate of $\theta$ is

$$\hat{\theta} = (b - a) \frac{\sum_{i=1}^{m} f(y_i)}{m},$$

where the $y_i$ are values of a random sample from a uniform distribution over $(a, b)$. 
Properties of the Estimator

The estimate is unbiased:

\[ E(\hat{\theta}) = (b - a) \frac{\sum E(f(Y_i))}{m} \]
\[ = (b - a)E(f(Y)) \]
\[ = \int_{a}^{b} f(x) \, dx. \]

The variance is

\[ V(\hat{\theta}) = (b - a)^2 \frac{\sum V(f(Y_i))}{m^2} \]
\[ = \frac{(b - a)^2}{m} V(f(Y)) \]
\[ = \frac{(b - a)}{m} \int_{a}^{b} \left( f(x) - \int_{a}^{b} f(t) \, dt \right)^2 \, dx. \]

The integral in the variance is a measure of the roughness of the function.
Importance Sampling

Suppose that the original integral can be written as

\[ \theta = \int_D f(x) \, dx = \int_D g(x)p(x) \, dx, \]

where \( p(x) \) is a probability density over \( D \).

Now, suppose that we can generate \( m \) random variates \( y_i \) from the distribution with density \( p \). Then, our estimate of \( \theta \) is just

\[ \tilde{\theta} = \frac{\sum g(y_i)}{m}. \]

The variance of \( \tilde{\theta} \) is likely to be smaller than that of \( \hat{\theta} \).

The use of a probability density as a weighting function also allows us to apply the Monte Carlo method to improper integrals.
Quadrature: Approximation or Estimation

Quadrature is an important topic in numerical analysis, and a number of quadrature methods are available.

The use of methods such as Newton–Cotes involves consideration of error bounds, which are often stated in terms of some function of a derivative of the integrand evaluated at some unknown point.

Monte Carlo quadrature differs from other numerical methods in a fundamental way: Monte Carlo methods involve random (or pseudorandom) sampling.

Instead of error bounds or order of the error as some function of the integrand, we use the variance of the random estimator to indicate the extent of the uncertainty in the solution.
Error Bounds and Uncertainty

The error bounds in the standard methods of numerical quadrature involve the dimension. They increase with increasing dimension.

The important thing to note from the expression for the variance of the Monte Carlo estimator is the order of error in terms of the Monte Carlo sample size; it is $O(m^{-1/2})$.

This results in the usual diminished returns of ordinary statistical estimators; to halve the error, the sample size must be quadrupled.

But it does not depend on the dimension!

(We should be aware of a very important aspect of a discussion of error bounds for the Monte Carlo estimators. It applies to random numbers. The pseudorandom numbers that we actually use only simulate the random numbers, so “unbiasedness” and “variance” must be interpreted carefully.)
Pricing Options by Monte Carlo

We will later consider some modifications to the pricing model that makes it more realistic.

However, in many cases the fair prices cannot be solved analytically from those models.

Pricing of various derivative instruments is an area in finance in which Monte Carlo methods can be used to analyze more realistic models.

First, let’s consider the simulation of the simple model.
Simulating Paths

Let’s simulate $m$ paths of the price of a stock over a period of time $t$.

We plot 100 simulated paths of the price of a stock for one year with $x(0) = 20$, $\Delta t = 0.01$, $\mu = 0.1$, and $\sigma = 0.2$. 
The price of the European call option should be the expected value of the payoff of the option at expiration discounted back to $t = 0$,

$$P(0, x) = e^{-qt_s} \mathbb{E}(h(X_{t_s})),$$

where $q$ is the rate of growth of an asset.

We estimate this expected value in the usual way.

(Although the Monte Carlo estimand in this case is an integral as before, it is an integral over a random path — that is, there is another source of variation. The multiple Monte Carlo estimators provide information about the distribution. This is an added bonus of the Monte Carlo approach.)

A problem is the choice of the rate of growth $q$.

A replicating strategy consistent with a no-arbitrage assumption leads us to a choice of $q = r$, the riskless rate.
Free Boundary Conditions

An American option could be priced by maximizing the expected value over all stopping times, $0 < \tau < t_s$:

$$P(0, x) = \sup_{\tau \leq t_s} e^{-q\tau} \mathbb{E}(h(X_\tau)).$$

(Of course, the sup is just a max.)
Parameters in the Model

In addition to the problem of choosing a correct models, in order to use the model we are still faced with the problem of estimating the parameters in the model.

As we have already seen, this is not as straightforward as it may seem.

The most important parameter is the volatility; that is, the standard deviation.

We could use a simulation-based approach to estimation of the parameters in the model (based on minimization of a measure of the difference in an ECDF and an SCDF, “simulated CDF”).

Let’s consider the volatility in more detail.
Volatility

As in all areas of science, our understanding of a phenomenon begins by identifying the quantifiable aspects of the phenomenon, and then our understanding is limited by our ability to measure those quantifiable aspects.

Most models of prices of financial assets have a parameter for volatility.

For a model to be useful, of course, we must have some way of supplying a value for the parameter, either by direct measurement or by estimation.

The volatility parameter presents special challenges for the analyst.
Volatility

Although often the model assumes that the volatility is constant, volatility, like most parameters in financial models, varies over time.

There may be many reasons for this variation in volatility including arrival of news.

When the variation is not explained by a specific event, the condition is known as “stochastic volatility”.

Empirical evidence supports the notion that the volatility is stochastic.

Volatility itself is a measure of propensity for change in time.

This results in the insurmountable problem of providing a value for a parameter that depends on changes of other values in time, but which itself changes in time.
Stylized Properties of Volatility

- Volatility is serially correlated in time

- Both positive news and negative news lead to higher levels of volatility

- Negative news tends to increase future volatility more than positive news

- There are two distinct components to the effect of news on volatility, one with a rapid decay and one with a slow decay

- Volatility has an effect on the risk premium
What Does the Black-Scholes Formula Tell Us about Volatility?

The Black-Scholes model and the resulting Black-Scholes formula include a volatility parameter, $\sigma$.

The motivation for the development of the Black-Scholes formula was to provide an approximate fair price for an option.

Another formula for the fair price of anything for which there is an active market, however, is the market price!
The Volatility Implied by the Black-Scholes Formula

For a given option (underlying, type, strike price, and expiry) and given the price of the underlying and the riskfree rate, the Black-Scholes formula relates the option price to the volatility of the underlying; that is, the volatility determines the model option price.

The actual price at which the option trades can be observed, however.

If this price is plugged into the Black-Scholes formula, the volatility can be computed.

This is called the “implied volatility”.

Ironically, we use the Black-Scholes formula to study volatility.
**Implied Volatility**

We should note that implied volatility may be affected by either the absence of “fairness” in the market price, or by the market price being correlated to Black-Scholes formulaic prices.

In the case of thinly traded assets, the market price may be strongly affected by idiosyncrasies of the traders.
Implied Volatility

Let $c$ be the observed price of the call.

Now, set $C_{BS}(t, S_t) = c$, and

$$f(\sigma) = S_t \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

We have

$$c = f(\sigma).$$

Given a value for $c$, that is, the observed price of the option, we can solve for $\sigma$.

There is no closed-form solution, but we can solve it iteratively.
Implied Volatility

As we have mentioned, one of the problems of the Black-Scholes formula is its assumption that the volatility $\sigma$ is constant.

If the volatility is constant, obviously if we substitute the observed market price of a particular option for the Black-Scholes price for that option, and do the same for a different option on the same underlying, we should get the same value.

We do not.
Implied Volatility

The implied volatility, for given $T$ and $S_t$, depends on the strike price, $K$.

In general, the implied volatility is greater than the empirical volatility, but the implied volatility is even greater for far out-of-the-money calls.

It also generally increases for deep in-the-money calls.

This variation in implied volatility is called the “smile curve”, or the “volatility smile”, as shown in the figure.
Implied Volatility

The curve is computed for a given security and the traded derivatives on that security with a common expiry.

The empirical price for each derivative at that expiry is used in the Black-Scholes formula and, upon inversion of the formula, yields a value for the volatility.
The Volatility Smile

The available strike prices are not continuous, of course.

The curve is a smoothed (and idealized) fit of the observed points.

The smile curve is not well-understood, although we have a lot of empirical observations on it.

Interestingly, prior to the 1987 crash, the minimum of the smile curve was at or near the market price $S_t$.

Since then it is generally at a point larger than the market price.

In addition to the variation in implied volatility at a given expiry, there is variation in the implied volatility at a given strike at different expiries.
Variation in Volatility over Time

The volatility also varies in time.

Volatility varies in time, with periods of high volatility and other periods of low volatility.

This is called “volatility clustering”.

The volatility of an index is somewhat similar to that of an individual stock.

The volatility of an index is a reflection of market sentiment. (There are various ways of interpreting this!)

In general, a declining market is viewed as “more risky” than a rising market, and hence, it is generally true that the volatility in a declining market is higher.

Contrarians believe high volatility is bullish because it lags market trends.
An Approach for Defining Market Volatility

A standard measure of the overall volatility of the market is the CBOE Volatility Index, VIX, which CBOE introduced in 1993 as a weighted average of the Black-Scholes-implied volatilities of the S&P 100 Index from at-the-money near-term call and put options. (“At-the-money” is defined as the strike price with the smallest difference between the call price and the put price.)

In 2004, futures on the VIX began trading on the CBOE Futures Exchange (CFE), and in 2006, CBOE listed European-style calls and puts on the VIX.

Other measures of the overall market volatility include the CBOE Nasdaq Volatility Index, VXN.
The VIX

The VIX initially was computed from the Black-Scholes formula, but now the empirical prices are used to fit an implied probability distribution, from which an implied volatility is computed.

In 2006, it was changed again to be based on the volatilities of the S&P 500 Index implied by several call and put options, not just those at the money, and it uses near-term and next-term options (where “near-term” is the earliest expiry more than 8 days away).

To compute the VIX, the CBOE uses the prices of calls with strikes above the current price of the underlying, up to the point at which two consecutive such calls have no bids. (Similarly for puts.)

The price of an option is the “mid-quote” price, i.e. the average of the bid and ask prices.
The VIX

The next figure shows the VIX for the period January 1, 1990, to December 31, 2007, together with the absolute value of the lograte returns of the S&P 500, on a different scale.

The peaks of the two measures correspond, and in general, the VIX and the lograte returns of the S&P 500 seem to have similar distributions.
SPX daily log returns

VIX daily closes

Jul 2003 to Jun 2008

2004 2005 2006 2007 2008
More Realistic Models

The fact that the implied volatility is not constant for a given stock does not mean that a model with an assumption of constant volatility cannot be useful.

It only implies that there are some aspects of the model that do not correspond to observational reality.

Another very questionable assumption in the simple model is that the changes in stock prices follow an i.i.d. normal distribution.

There are several other simplifications, such as the restriction to European options, the assumption that the stocks do not pay dividends, the assumption that we can take derivatives as if $a_t$ were constant, and so on.

All of these assumptions allow the derivation of a closed-form solution.
More Realistic Models

There are various approaches for modifying the simple models.

Instead of an SDE with a single stochastic differential component for continuous diffusion, we may incorporate a second stochastic differential component that represents Poisson jumps.

Instead of an SDE with a single stochastic differential component, we can use a couple system of SDEs, one for the standard deviation of the other one.

There are various ways of modifying the basic autoregressive model. One modification is generalized autoregressive conditional heteroskedasticity (GARCH).
Fixing the Models: Coupled Diffusion

A simple model in which the volatility is a function of a separate mean-reverting Ornstein-Uhlenbeck process:

\[ dX_t = \mu X_t dt + \sigma_t X_t dB_t, \]
\[ d\sigma_t^2 = f(Y_t), \]
\[ dY_t = \alpha(\mu_Y - Y_t) dt + \beta d\tilde{B}_t, \]

where \( \alpha \) and \( \beta \) are constants and \( \tilde{B}_t \) is a linear combination of \( B_t \) and an independent Brownian motion.

The function \( f \) can incorporate various degrees of complexity, including the simple identity function.

These coupled equations provide a better match for observed stock and option prices.
Fixing the Models: Adding Jumps to Diffusion

A very realistic modification of the model is to assume that $Z$ has a superimposed jump or shock on its $N(0, 1)$ distribution.

A useful model is

$$dX_t = \mu_t dt + \sigma_t dW_t + \kappa_t dq_t,$$

where $dq_t$ is a Poisson process with intensity $\lambda_t$; that is, $\Pr(dq_t = 1) = \lambda_t dt$.

We may use a constant value for $\kappa_t$ or else some reasonable distribution, say a normal with a negative mean.

A preponderance of bear jumps, that is, instances in which $\kappa_t < 0$, of course, would decrease the fair price of a call option from the Black–Scholes price and would increase the fair price of a put option.
Fixing the Models: ARIMA-Type Models with Heteroscedasticity

If the innovations in the time series model are not i.i.d., various other models may be more appropriate.

One is the generalized autoregressive conditional heteroscedastic (GARCH) model,

\[ X_t = E_t \sigma_t, \]

where the \( E_t \) are i.i.d. \( \text{N}(0, 1) \),

\[ \sigma_t^2 = \sigma^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-q}^2, \]

\[ \sigma > 0, \ \beta_i \geq 0, \ \alpha_i \geq 0, \]

and

\[ \sum_{i=1}^{p} \beta_i + \sum_{j=1}^{q} \alpha_j < 1. \]

A GARCH(1,1) model is often adequate.
Non-Gaussian Randomness

The Gaussian distributions used in these models could be replaced with heavier-tailed distributions.

The Brownian motion (Gaussian) in the simple diffusion, the coupled diffusion, or the jump-diffusion models could be replaced with some other distribution.

In the GARCH models the assumption of normality for the $E_t$ can easily be changed to some other distribution.

A GARCH with innovations following a Student’s $t$ distribution with finite degrees of freedom (“TGARCH”)

A stable distribution other than the normal (“SGARCH”).

These models are easy to simulate with various distributions.
Using More Realistic Models

More realistic models can be studied by Monte Carlo methods, and this is currently a fruitful area of research.

Paths of prices of the underlying can be simulated using the coupled model, as we did with the simpler model.

We can use different distributions on $Z$.

The simulated paths of the price of the underlying provide a basis for determining a fair price for the options.

This price is just the break-even value discounted back in time by the risk-free rate $r$.

Other modifications to the underlying distribution of $\Delta X_t$ result in other differences in the fair price of options.
Research Topics: Statistical Modeling

There are many fruitful areas of research in computational finance.

Computational statisticians with very little background in finance or economics, because of fresh perspectives, can often make useful contributions.

Statistical modeling always presents a wide range of research topics, and financial modeling provides a wide range of opportunities, especially when covariate modeling of multiple streams of data is contemplated.

There are many open issues concerning volatility, especially its temporal variability.

Even the measurement of volatility is an issue. Does the VIX directly relate to any of the parameters in our pricing models?
There are some statistical tests for the existence of jumps, but they are either anti-conservative or lack any meaningful power.

Monte Carlo tests may be of some use.

The model parameters are not directly observable; rather, they must be derived from observable data.

Although the models are still very approximate, in some ways the modeling has advanced faster than the statistical methods for estimating the parameters in the models.

Monte Carlo simulation, of course, allows for quick preliminary investigation of ideas.
Research Topics: Data Mining

There are many challenges in financial data mining.

The integration of data of various types is perhaps the primary challenge in mining financial data.

A major obstacle to the relevance of any financial data is the fact that the phenomena measured by or described by the data are changing over time.

The primary challenges in the mining of financial data arise from the vast diversity of types and sources of data and from the variation of the data in time.