CSI972 Midterm
In class
Sample
One sheet of pre-written notes

Write your solutions on these pages. Be sure to write clearly and to show all steps. Be sure to put your name on your exam.

1. Given probability spaces $(\Omega, \mathcal{F}, P)$ and $(\mathrm{I\!R}, \mathcal{B}, \nu)$ where $\mathrm{I\!R}$ are the reals and $\mathcal{B}$ is the Borel σ-field; the sets $A_1, A_2, \ldots \in \mathcal{F}$; and the random variables X and Y from Ω into $\mathrm{I\!R}$.

   (a) Write out $\sigma(\{A_1, A_2\})$.

   (b) Assume $\sum_{i=1}^{\infty} P(A_i) < \infty$. Show that $P(\limsup_i A_i) = 0$.

   (c) Write out $\sigma(\sigma(\{A_1\}) \times \sigma(\{A_2\}))$.

   (d) Let λ be the Lebesgue measure. Find

      i. $\lambda([1/2, 3/4))$.

      ii. $\lambda(\cup_{i=1}^{\infty}(\frac{1}{2^{i+1}}, \frac{1}{2^i}])$.

      iii. Let $f(x) = c\mathrm{I}_{[a,b)}(x)$. Under what conditions, if any, is f a density w.r.t. λ?

   (e) If $\Omega = [0, 1]$, $X(\omega) = 2\omega$, and $Y(\omega) = \omega^2$ for $\omega \in \Omega$, find $\mathrm{E}(X|Y)$.

2. Let $X_1, X_2, \ldots$ be a sequence of random variables and let $\overline{X}_n = \sum_{i=1}^{n} X_i/n$. Show that if $X_n \rightarrow_{a.s.} 0$, then $\overline{X}_n \rightarrow_{a.s.} 0$.

3. Suppose $X_1, \ldots, X_n$ are independent random variables with the same density in the exponential family:

$$p_\theta(x) = \exp\left((\eta(\theta))^{\mathrm{T}} T(x) - \xi(\theta)\right) h(x).$$

(a) Show that the joint distribution of X_1 and X_2 is in an exponential family.

(b) Show that the distribution of $Y = X_1 + X_2$ is in an exponential family.

(c) Determine $\mathrm{E}(\overline{X})$, where $\overline{X} = \sum_{i=1}^{n} X_i/n$.

(d) Determine $\mathrm{V}(\overline{X})$.

4. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{U}\left(\theta - \frac{1}{2},\ \theta + \frac{1}{2}\right)$.
Let $T = \left(X_{(1)},\ X_{(n)}\right)$.

(a) Show that T is sufficient.

(b) Show that T is minimal sufficient.

(c) Show that T is not complete.

5. Consider the basic problem in decision theory: we have a probability space $(\Omega, \mathcal{F}, P)$ with $P \in \mathcal{P}$. We have an "action space" $\mathcal{A}$, which consists of decisions about which $P \in \mathcal{P}$ corresponds to an observable random variable X. We base inference on the random variable and the action is a mapping, T, from $\mathcal{X}$, the range of X, to $\mathcal{A}$. If we observe X, we take the action $T(X) = a \in \mathcal{A}$. Given a σ-field $\mathcal{F}_{\mathcal{X}}$ over $\mathcal{X}$ and a σ-field $\mathcal{F}_{\mathcal{A}}$ over $\mathcal{A}$, we require that T be measurable $\mathcal{F}_{\mathcal{X}}/\mathcal{F}_{\mathcal{A}}$.

   Our interest will be in finding an "optimal" decision rule T.

   Outline a systematic approach to this problem. Define the quantities that are relevant to your approach.