The Black-Scholes Differential Equation
Delta Hedging and Continuous Dividends

Recall two weeks ago we developed the B-S DE by a "replicating portfolio" consisting of the underlying, the option, and cash. We mentioned at that time that we could have based the derivation on a delta hedging strategy. We did not go through the details then. I will do so now, but I will also add a wrinkle that we have been ignoring: dividends.

A delta hedge is simple. Let $V(S,t)$ be the value of one option contract and let $\Delta = V(S,t)/S$. Thus, a fully hedged portfolio is

$$\Pi = V(S,t) - \Delta S,$$

and its change is

$$\mathrm{d}\Pi = \mathrm{d}V(S,t) - \Delta \mathrm{d}S.$$

Note: Δ is not an operator here; it is just a number.

The underlying follows a lognormal random walk (assumption), and by Ito's formula we have (dropping the (S,t) in V)

$$\mathrm{d}V = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\mathrm{d}t,$$

yielding

$$\mathrm{d}\Pi = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\mathrm{d}t - \Delta\mathrm{d}S.$$

The randomness, that is, the terms involving $\mathrm{d}S$ is reduced to zero if we choose

$$\Delta = \frac{\partial V}{\partial S}.$$

(Notice S is not instantaneously random, but $\mathrm{d}S$ is.)

This is where the drift μ goes. It is assumed into Δ.

Now what is Δ? It may change with time; hence this hedged portfolio must be dynamic (just as our replicating portfolio was in the previous development).

With this choice of Δ, we have

$$\mathrm{d}\Pi = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\mathrm{d}t.$$

Now, if we assume no arbitrage, we have

$$\mathrm{d}\Pi = r\Pi\mathrm{d}t,$$

and then with the appropriate substitutions and dividing by $\mathrm{d}t$, we have the Black-Scholes differential equation as before,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

A purpose of the foregoing was to see the previous development from a slightly different perspective.

Another purpose is to show how we could introduce the effect of dividends.

Dividends are actually rather difficult to incorporate into the model for two reasons; they are not guaranteed (they are stochastic, in some sense), and they are lumpy.

We will make two simplifying assumptions: dividends are deterministic with a constant rate (at least locally), and they are continuously accruing.

Assume dividends can be modeled as a continuously-compounded payout at the rate q; hence an amount ΔS increases in time as

$$q\Delta S \mathrm{d}t.$$

Although this is not the case at all, we could argue that knowledgable traders increase the price in this fashion.

Now, if we put this into our formula for the hedged portfolio, $\Pi = V - \Delta S$, we have

$$\mathrm{d}\Pi = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\mathrm{d}t - \Delta \mathrm{d}S - q\Delta S \mathrm{d}t.$$

Now, again we let $\Delta = \partial V/\partial S$, yielding

$$\mathrm{d}\Pi = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\mathrm{d}t - q\Delta S\mathrm{d}t,$$

and finally setting this to the risk-free change $r\Pi\mathrm{d}t$, and dividing by $\mathrm{d}t$ we have the Black-Scholes differential equation with continuously compounded dividends,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0.$$

Similar methods can be used to arrive at Black-Scholes equations for other types of options, including currency options in which there is an interest rate on the foreign currency, commodity options in which there are carrying costs, and options on futures (which actually is simpler than the standard DE for stock options).

The Black-Scholes Formulas In the Presence of Continuous Dividends

We can solve the Black-Scholes equation with *continuously-compounded* dividends as we did last week for the equation without dividends. We first apply the boundary condition and then we recognize the normal integrals and identify the appropriate normal means and standard deviations. The formula for the European-style call is

$$C_{\mathsf{BS}}(t,S) = Se^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{1}$$

where now

$$d_1 = \frac{\log(S/K) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and, as before

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Notice that the dividend rate q is used to discount the price of the underlying, just as the interest rate r is used to discount everything.

The formula for European-style puts can be derived in the same way by applying the appropriate boundary condition.

The Greeks

There are a number of important relationships between the value of an option and the individual variables or parameters that determine its price.

Since most of these are represented by Greek letters, they are called "greeks".

Delta, Gamma, Vega, Theta, Rho

We have encountered Delta before in a "delta hedge".

The greeks are used in various hedging strategies.

Delta

For the rate of change of value V of a portfolio of options (usually just a single option) on a single underlying with price S, as a function of that price,

$$\Delta = \frac{\partial V}{\partial S}.$$

Gamma

For the rate of change of the rate of change of value V of a portfolio of options (usually just a single option) on a single underlying with price S, as a function of that price,

$$\Gamma = \frac{\partial^2 V}{\partial S^2}.$$

Theta

For the rate of change of value V of a portfolio of options (usually just a single option), as a function of time,

$$\Theta = \frac{\partial V}{\partial t}.$$

Vega

(Vega is not a Greek letter. It is the name of a star. But Vega is also called kappa, κ (lower case, because the upper case kappa is just K.))

For the rate of change of value V of a portfolio of options (usually just a single option), as a function of the (constant!) volatility,

$$\text{Vega} = \frac{\partial V}{\partial \sigma}.$$

Rho

(We write this as ρ (lower case, because the upper case rho is R, which looks like a Latin R.))

For the rate of change of value V of a portfolio of options (usually just a single option), as a function of the (constant!) risk-free interest rate,

$$\rho = \frac{\partial V}{\partial r}.$$

Formulas for the Greeks

Because we have closed form solutions for the Black-Scholes differential equation, we can get closed form solutions for the greeks under those assumptions. For example, in the notation above for a European-style call on an underlying that pays continuously-compounded dividends with rate q, we have

$$
\begin{aligned}
\Delta &= \mathrm{e}^{-q(T-t)}\Phi(d_1) \\
\Gamma &= \frac{\phi(d_1)\mathrm{e}^{-q(T-t)}}{S\sigma\sqrt{T-t}} \\
\text{Vega} &= S\sqrt{T-t}\,\phi(d_1)\mathrm{e}^{-q(T-t)} \\
\Theta &= -\frac{S\phi(d_1)\sigma\mathrm{e}^{-q(T-t)}}{2\sqrt{T-t}} - rK\mathrm{e}^{-r(T-t)}\Phi(d_2) + qS\mathrm{e}^{-q(T-t)}\Phi(d_1) \\
\rho &= K(T-t)\mathrm{e}^{-r(T-t)}\Phi(d_2)
\end{aligned}
$$

where $\phi(x)$ is the normal PDF; that is, $\phi(x) = \Phi'(x)$.

**** Exercise: derive the formulas for European-style puts.

Another Approach to Pricing Options

The approach to pricing options that we have followed so far depended on a stochastic model for prices, the concept of a hedging portfolio, and deflated prices based on a numeraire.

This approach leads to an SDE.

If the model is simple enough, we can get a closed form solution.

The things that made it simple were
- European style options on a single underlying
- constant normal distribution of price changes.

Going beyond this simple case, we get complicated DEs that are not of much use. We need a different approach.

Another approach is to formulate the prices as expected values.

Expected values can be estimated by Monte Carlo methods.

Expected Value of Prices

As before, let T be the time of exercise, $S(t)$ is the price of the underlying at time t, and $V(S,t)$ is the value of the option at time t.

It would seem reasonable that $V(S,t)$ should relate to the expected value $\mathsf{E}(V(S,T))$ discounted back to time t.

A fixed discount rate, however, would not account for the risk.

We need a stochastic discount factor, which like the risk free rate, applies to any assets.

Our objective is a martingale, that is, something such that

$$\mathsf{E}_Q\left(\eta(T)V(T)\,|\,\mathcal{F}_{\mathcal{Q}}\right) = \eta(t)V(t).,$$

under some probability measure Q.

Now that we have an idea of the objective, let's write this in a more convential form (similar to Tavella on pages 60 and 61).

We introduce a stochastic discount factor, $Z(t)$, which is strictly positive, and such that, for any *attainable* price process $X(t)$,

$$\frac{X(t)}{Z(t)} = \mathsf{E}_Q\left(\frac{X(T)}{Z(T)} \,|\, \mathcal{F}_{\mathcal{Q}}\right).$$

("Attainable" means that securities exist and can be combined in such a way as to sum to 0.)

This discount factor will serve as the fundamental measure or numeraire for all attainable price processes.

Now, if we are interested in the value $V(t)$ at a particular time t_0, we can normalize $Z(t)$ so that $Z(t_0) \equiv 1$, we have

$$V(t_0) = \mathsf{E}_Q\left(\frac{V(T)}{Z(T)} \,|\, \mathcal{F}_{\mathcal{Q}}\right). \tag{2}$$

A concept we *assumed* and used before was that of a risk-free rate of return r and a strictly positive risk-free asset $\beta(t)$. The assumption is that there is an attainable price process $\beta(t)$ such that $\mathrm{d}\beta(t)/\beta(t) = r\mathrm{d}t$.

Does this exist? Does the stochastic discount factor $Z(t)$ exist?

The questions are related.

The relationship between the strictly positive processes $\beta(t)$ and $Z(t)$ allows us to define a new measure P_β in terms of the measure Q associated with $Z(t)$:

$$\frac{\mathrm{d}P_\beta}{\mathrm{d}Q} = \frac{\beta(t)}{Z(t)}, \quad \text{for } 0 \le t \le T.$$

(The measure Q is sometimes called an "objective measure". Its role is to relate "real world" probabilities to the geometric Brownian motion SDE as it evolves in time.)

The Basis for Monte Carlo Simulation

The relationship between P_β and Q means that for any $A \in \mathcal{F}_\mathcal{Q}$ means that we can express $P_\beta(A)$ as

$$\begin{aligned} P_\beta(A) &= \mathsf{E}_Q\left(\mathrm{I}_A\left(\frac{\mathsf{d}P_\beta}{\mathsf{d}Q}\right)\right) \\ &= \mathsf{E}_Q\left(\mathrm{I}_A\left(\frac{\beta(t)}{Z(t)}\right)\right), \end{aligned}$$

where $\mathrm{I}_A(\cdot)$ is the indicator function.

This gives us for the expectation with respect to the measure P_β for any nonegative $X(t)$ that is measureable with respect to $\mathcal{F}_\mathcal{Q}$,

$$\mathsf{E}_\beta(X(t)) = \mathsf{E}_Q\left(X(t)\frac{\beta(t)}{Z(t)}\right)$$

... continued

Now, combining this with equation (2), we have equation (5.1) in Tavella, with $\beta(t_0) = 1$,

$$
\begin{aligned}
V(t_0) &= \mathsf{E}_\beta\left(\frac{V(T)}{\beta(T)}\right) \\
&= \mathrm{e}^{-r(T-t_0)}\mathsf{E}_\beta(V(T)). \qquad (3)
\end{aligned}
$$

Equation (3) is the basis for using Monte Carlo simulation for in asset pricing.

Before proceeding, let's look back at some of the things we have done.

Fundamental Theorem of Asset Pricing

The question is whether such a stochastic discount factor $Z(t)$ exists.

The answer is that it does exist under the assumption of the existence of risk-free asset that grows at a fixed rate.

A more remarkable fact (under a few technical assumptions, including again the existence of a fixed risk-free rate) the no-arbitrage condition implies the existence of a stochastic discount factor $Z(t)$ and conversely, the existence of a stochastic discount factor $Z(t)$ implies the no-arbitrage condition.

This equivalence is sometimes called the Fundamental Theorem of Asset Pricing.

It depends on some theory that is beyond the scope of this course.

... more comments

Equation (3) gives us the current value (at time t_0) as the expected present value of the terminal value $V(T)$ *discounted at the risk-free rate* r, rather than at the stochastic discount factor $Z(t)$.

Furthermore, the probability measure of interest is P_β rather than Q. P_β is called the risk-neutral measure.

Equation (3) is important because the dynamics of $Z(t)$ are generally unknown and would be difficult to model in any event.

... more comments

There is one final implication of the foregoing to note.

In the original geometric Brownian motion model, the probability measure is the Q that we have been using, so we might write it as

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W^{Q}(t)$$

to emphasize that fact. We have now developed an equivalent measure P_β under which attainable prices in terms of $\beta(t)$ are martingales; hence, we have

$$\frac{\mathrm{d}S(t)}{\beta(t)} = \frac{S(t)}{\beta(t)} \sigma \mathrm{d}W^{P_\beta}(t),$$

where $\mathrm{d}W^{P_\beta}(t)$ is a Brownian motion with respect to P_β. For notational simplicity, since it is what we will be using henceforth, let's just write it as $\mathrm{d}W(t)$.

... continued

We can relate $\mathrm{d}W(t)$ and $\mathrm{d}W^Q(t)$ by some process $\nu(t)$ satisfying

$$\mu = r + \sigma\nu;$$

that is,

$$\mathrm{d}W(t) = \mathrm{d}W^Q(t) + \nu(t).$$

We then have

$$\frac{\mathrm{d}S(t)}{S(t)} = r\mathrm{d}t + \sigma\mathrm{d}W(t). \tag{4}$$

This is what we need for Monte Carlo simulation; the individual drift μ is not present! (If it were, we could not expect to get anything that corresponds to the Black-Scholes differential equation.)

Monte Carlo Methods

Monte Carlo pricing involves the estimation of $V(t)$ in equation (3) using equation (4).

For given t_0, $S(t_0)$, K, and T, we simulate a path for the price of the underlying, and compute the simulated payoff. The average payoff for many simulation runs is the estimate of the value at time T, so the estimate of the value at time t_0 is just the estimate of the final value discounted by r back to t_0.

How do we simulate the payoff of a given path; that is, how do we simulate the value at the end of the path?

By the properties of Brownian motion, the value at the end of the path depends only on the length of the path; that is, we only need one random step!

So we have

$$S(T) = S(t)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}Z\right),$$

where $Z \sim \mathrm{N}(0,1)$. (See lectures 1 and 2.)

Doing this in a Monte Carlo sample of size m, with $Z_1, \ldots, , Z_m$, we have as an estimate of the appropriate price of a European-style call on an underlying that pays no dividends,

$$C_{\mathrm{MC}} = \mathrm{e}^{-r(T-t)}\frac{1}{m}\sum_{i=1}^{m}\left(S(t)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}Z_i\right) - K\right)^+. \quad (5)$$

(Compare equation (5) from lecture 10.)

In similar fashion we could get a formula for the Monte Carlo estimate of the present value of a put.

That's too easy. So what's the big deal?

Are μ and σ constant in time? How do they vary? stochastically? As an Ito process?

American-style options.
Bermudan options.
Asian options.

The basic simulation of a Brownian motion is simple for particular points in time, $0 = t_0 < t_1 < t_2 < \cdots$:

$$W(t_i) = W(t_{i-1}) + \sqrt{t_i - t_{i-1}} Z_i,$$

where the Zs are iid $\mathrm{N}(0, 1)$, and $W(0) = 0$.