Semester Projects

Pricing Derivatives: Basics

Start with a European call option. How do we value it?

Current time: t_0

Expiration: T

Strike price: K

Price of underlying: $S(t_0), \ldots, S(t), \ldots, S(T)$

Value of the call: $C(t_0), \ldots, C(t), \ldots, C(T)$
P for put; V for either

We have been (and in this course will continue to be) vague about the pricing unit. In general, we call the price, or the pricing unit, a *numeraire*. A more careful development of this concept rests on the idea of a *pricing kernel*.

We will usually call them dollars.

The Price of a European Call Option

A *European call option* is a contract that gives the owner the right to buy a specified amount of an underlying for a fixed *strike price*, K on the *expiration* or *maturity* date T.

The owner of the option does not have any obligations in the contract.

The *payoff*, h, of the option at time T is either 0 or the excess of the price of the underlying $S(T)$ over the strike price K.

Once the parameters K and T are set, it is a function of $S(T)$:

$$h(S(T)) = \begin{cases} S(T) - K & \text{if } S(T) > K \\ 0 & \text{otherwise} \end{cases}$$

The Price of Call Options

The price of the option at any time is a function of the time t, and the price of the underlying s. We denote it as $P(t, s)$.

What is the price at time $t = 0$?

It seems natural that the price of the European call option should be the expected value of the payoff of the option at expiration, discounted back to $t = 0$:

$$P(0, s) = e^{-rT} \mathsf{E}(h(S(T))).$$

Likewise, for an American option, we could maximize the expected value over all stopping times, $0 < \tau < T$:

$$P(0, s) = \sup_{\tau \leq T} e^{-r\tau} \mathsf{E}(h(S(\tau))).$$

Principles

Basic principle of Black-Scholes: We seek a portfolio with zero expected value, that consists of short and/or long positions in the option, the underlying, and a risk-free bond.

There are two key ideas in developing pricing formulas for derivatives:

1. no-arbitrage principle

2. replicating, or hedging, portfolio

An *arbitrage* is a trading strategy with a guaranteed rate of return that exceeds the riskless rate of return.

In financial analysis, we assume that arbitrages do not exist.

Example of the No-Arbitrage Principle

Consider a forward contract that obligates one to pay K at T for the underlying. At time t, with $t < T$, the price of the underlying is $S(t)$. What should the price of the contract be or, equivalently,

What should K be so that the price of the contract is 0?

Its value at expiry is $S(T) - K$, and of course we do not know $S(T)$. If we have a riskless rate of return r, we can use the no-arbitrage principle to determine the correct price of the contract.

To apply the no-arbitrage principle, consider the following strategy:

take a long position in the forward contract and
sell the underlying short.

With this strategy, the investor immediate receives $S(t)$. At time T this amount can be guaranteed to be

$$S(t)e^{r(T-t)}.$$

The No-Arbitrage Principle

If

$$K < S(t)e^{r(T-t)},$$

a long position in the forward contract and
a short position in the underlying

is an arbitrage.

Conversely, if

$$K > S(t)e^{r(T-t)},$$

a short position in the forward contract and
a long position in the underlying

is an arbitrage.

Therefore, under the no-arbitrage assumption, the correct value of K is $S(t)e^{r(T-t)}$.

The replication approach is to determine a portfolio and an associated trading strategy that will provide a payout that is identical to that of the underlying. This portfolio and trading strategy *replicates* the derivative.

A replicating strategy involves both long and short positions.

Pricing Derivatives

If every derivative can be replicated by positions in the underlying (and cash), the economy or market is said to be *complete*.

We will generally assume complete markets.

The Black-Scholes approach leads to the idea of a self-financing replicating hedging strategy.

The approach yields the interesting fact that the price of the call does not depend on the expected value of the underlying. It does depend on its volatility, however.

Self-Financing Replicating Hedging Strategy

Neil Chriss's example of a casino.

Game is to flip a fair coin 3 times.

Casino pays $1 if heads occurs 3 times in a row, HHH.

How much should it cost to play? (Casino will then add operating and profit margin.)

Relation to options; who's short and who's long

Big player: $100,000.

Bet broker: casino bets $12,500 on H on first toss; casino is even

If H occurs, casino has $25,000 and bets on H on second toss

Analysis of Casino Hedging Example

Hedging – in general

Special properties:

Self-financing

Replicating.

Roles of three participants; who's short and who's long

bid-ask spread (will broker charge the expected value of the bet?)

other transaction costs...

Assumptions: market impact, complete market

Expected Rate of Return on Stock

XYZ selling at $S(t_0)$; no dividends.

What is its expected value at time $T > t_0$?

It is merely the forward price for what it could be bought now.

Forward price: $\mathrm{e}^{r(T-t_0)}S(t_0)$,
where r is the risk-free rate of return, $S(t_0)$ is the spot price, and $T - t_0$ is the time interval.

This is the no-arbitrage principle.

The expected value of the stock does not depend on the rate of return of the stock
(that's μ in some of the models we've used).

This is true because of the cost of a forward contract.

Call Option

Holder of the forward contract (long position) on XYZ must buy stock at time T for $e^{r(T-t_0)}S(t_0)$.

Holder of a call option buys stock only if $S(T) > K$.

Role of volatility
on forward contract holder (volatility not good)
on call option holder (volatility good) – enhances the value of the option

Assumptions ...

Conclusion under assumptions: expected volatility of the underlying affects the value of an option, but expected rate of return of the underlying does not.

Hedging

Hedging risk: refray the risk of one investment by making an offsetting investment.

Cost of hedge.

Return of hedge (reduced risk).

Perfect hedge: returns *exactly* the amount needed to cover any loss, and no more.

Example: write call (i.e., go short). what is a hedge? owning enough stock to cover is a hedge, but not perfect (it costs too much, and writer is not protected against drop in price of underlying).

Hedging

Black-Scholes approach constructs a portfolio consisting of some underlying and some risk-free (zero-coupon) bonds to offset the short call.

(discuss dividends, coupons, etc.)

Hedging

In a perfect hedge, the hedging instrument behaves exactly the way the hedged instrument does.

Call option vs. hedging instrument: value at T.

they're equal ... called "payoff replication".

Dynamic Hedging

process of managing the risk of options

hedging portfolio's value at any time is equal to the value of the option at that point in time.

1. replicates the payoff

2. has fixed and known total cost

weighted portfolio, balancing

hedging strategy produces a *synthetic* version of the option.

Self-Financing Dynamic Hedging

Cost of hedge:
1. infusion of funds cost
2. transaction costs (bid-ask, friction, inability to execute trades at exactly the price specified by the strategy, etc.)

A hedging strategy is self-financing if its total to-date cost at any time (excluding transaction costs) is equal to the setup cost.

Setup cost: initial outlay; e.q. strategy requires going long \$1,000 and short \$700; setup is \$300

To do this, we need a formula for the relative rates of change of the price of the call and that of the underlying,

$$\Delta = \frac{\mathrm{d}C}{\mathrm{d}S}.$$

Delta of an Option

The *delta of an option* is the rate of change of the option's *value* with respect to the change in the underlying's price.

Consider times t_0 and t_1. If the value of an option at time t is $V(t)$, and the price of the underlying is $S(t)$, the delta at t_0 is **approximated** by

$$\Delta_{t_0} = \frac{V(t_0) - \mathrm{e}^{-r(t_1 - t_0)} V(t_1)}{S(t_0) - \mathrm{e}^{-r(t_1 - t_0)} S(t_1)}.$$

We essentially neutralize the change in time by the risk-free rate.

comments: zero in denominator; time value

Market Models for Derivative Pricing

A simple model of the market assumes two assets:
a *riskless asset* with price at time t of β_t,
and a *risky asset* with price at time t of $S(t)$.

The price of a derivative can be determined based on trading strategies involving these two assets.

The price of the riskless asset follows the deterministic ordinary differential equation

$$\mathrm{d}\beta_t = r\beta_t \mathrm{d}t,$$

where r is the instantaneous riskfree interest rate.

The price of the risky asset follows the stochastic differential equation

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W_t.$$

Preliminary Formula

$$C(t) = \Delta_t S(t) - \mathrm{e}^{r(T-t)} B(t),$$

where $B(t)$ is the current value of a riskless bond.

We speak of a *portfolio* as a vector p whose elements sum to 1.

The length of the vector is the number of assets in our universe. Sometimes we limit this to assets whose values are independent of each other; that is, we may exclude derivatives.

The no-arbitrage principle can be stated as:
There does not exist a p such that for some $t > 0$, either

- $p^{\mathrm{T}} s_0 < 0$ and $p^{\mathrm{T}} S(t)(\omega) \geq 0$ for all ω,

or

- $p^{\mathrm{T}} s_0 \leq 0$ and $p^{\mathrm{T}} S(t)(\omega) \geq 0$ for all ω, and $p^{\mathrm{T}} S(t)(\omega) > 0$ for some ω.

A derivative D is said to be *attainable* (over a universe of assets $S = (S^{(1)}, S^{(2)}, \ldots, S^{(k)})$) if there exists a portfolio p such that for all ω and t,

$$D_t(\omega) = p^{\mathsf{T}} S(t)(\omega).$$

Not all derivatives are attainable. The replicating portfolio approach to pricing derivatives applies only to those that are attainable.

Dynamic and Self-Financing Portfolios

The value of a derivative changes in time and as a function of the value of the underlying; therefore, a replicating portfolio must be changing in time or “dynamic”.

In analyses with replicating portfolios, transaction costs are ignored.

Also, the replicating portfolio must be self-financing; that is, once the portfolio is initiated, no further capital is required. Every purchase is financed by a sale.

A Replicating Strategy

Using our simple market model, with
a *riskless asset* with price at time t of β_t,
and a *risky asset* with price at time t of $S(t)$,
(with the usual assumptions on the prices of these assets),
we can construct a portfolio whose value will almost surely the payoff of a European call option on the risky asset at time T.

At time t, the portfolio consists of a_t units of the risky asset, and of b_t units of the riskless asset. Therefore, the value of the portfolio is $a_t S(t) + b_t \beta_t$. If we scale β_t so that $\beta_0 = 1$ and adjust b_t accordingly, the expression simplifies, so that $\beta_t = e^{rt}$.

The portfolio replicates the value of the option at time T if it has value $K - S(T)$ if this is positive and zero otherwise.

If the portfolio is self-financing

$$\mathrm{d}(a_t S(t) + b_t e^{rt}) = a_t \mathrm{d}S(t) + r b_t e^{rt} \mathrm{d}t.$$