Mathematical Modeling of Large Deformations on a Non-Linear Plate

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Rigid Wing
Rigid Wing
Multiple Rotor

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Mathematical Modeling of Large Deformations on a Non-Linear Plate
Multiple Membrane Rotor

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Biomimicry

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Mathematical Modeling of Large Deformations on a Non-Linear Plate
First, we present the development of the mathematical model for the dynamic behavior of a nonlinear plate undergoing deformation both in transverse and axial directions using a Hamiltonian approach.
We use the Kirchhoff hypothesis for the deformation \((u_i)\) of the plate.

**Assumptions**

- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation

**Equations**

\[
\begin{align*}
    u_1 &= u - x_3 w_{x_1} \\
    u_2 &= v - x_3 w_{x_2} \\
    u_3 &= w
\end{align*}
\]

**Terms**

- \(u\) = axial displacement in the \(x_1\) direction
- \(v\) = axial displacement in the \(x_2\) direction
- \(w\) = transverse displacement
This is an example from a beam model, which we will ultimately compare our model against:

[Hickman (2010)]
This is what we are actually working with:

\[ \phi \approx \frac{dw}{dx_1} \]

- \( u \) is axial displacement in \( x_1 \) direction.
- \( v \) is axial displacement in \( x_2 \) direction.
- \( w \) is transverse displacement.

\[ \theta \approx \frac{dw}{dx_2} \]
We use the Green strain tensor to relate strain ($E_{ij}$) and displacement as follows:

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right)$$

$$E_{11} = \frac{1}{2} \left( 2(u_{x_1} - x_3 w_{x_1 x_1}) + w_{x_1}^2 \right)$$
$$E_{22} = \frac{1}{2} \left( 2(v_{x_2} - x_3 w_{x_2 x_2}) + w_{x_2}^2 \right)$$
$$E_{12} = E_{21} = \frac{1}{2} \left( u_{x_2} + v_{x_1} - 2x_3 w_{x_1 x_2} \right) + w_{x_1} w_{x_2}$$
We use the a materially linear formulation to relate stress ($\sigma_{ij}$) and strain using Young’s modulus ($Y$) and a Poisson ratio ($\nu$) as follows:

\[
\sigma_{11} = \frac{Y}{(1-\nu^2)}(E_{11} + \nu E_{22}) \\
\sigma_{22} = \frac{Y}{(1-\nu^2)}(E_{22} + \nu E_{11}) \\
\sigma_{12} = \sigma_{21} = \frac{1-\nu}{2} \frac{Y}{(1-\nu^2)} E_{12}
\]
Kinetic Energy

For a homogeneous plate density $\rho$, to account for all the mass $T$ takes the form of an integral over the area. We also drop the inertial term.

$$T = \frac{1}{2} m ||V||^2$$

$$T = \int_{a}^{a} \int_{0}^{a} \int_{\frac{h}{2}}^{\frac{h}{2}} \rho \left( \frac{\partial u_1}{\partial t} \right)^2 + \frac{\partial u_2}{\partial t} \left( \frac{\partial u_3}{\partial t} \right) \right) dx_3 dx_2 dx_1$$

Terms

$$u_1 = u - x_3 w_{x_1}$$
$$u_2 = v - x_3 w_{x_2}$$
$$u_3 = w$$

$$T = \int_{0}^{a} \int_{0}^{a} \frac{\rho h}{2} (u_t^2 + v_t^2 + w_t^2) dx_2 dx_1$$
The remaining potential energy is similar to a compressed spring, as follows:

\[
U = \int_0^a \int_0^a \int_{-h/2}^{h/2} \frac{1}{2} \left( \sigma_{11} E_{11} + \sigma_{22} E_{22} + \sigma_{12} E_{12} \right) dx_3 dx_2 dx_1
\]

\[
= \frac{Y}{(1 - \nu^2)} \int_0^a \int_0^a h \left( (u_{x_1} + \frac{1}{2} w_{x_1}^2)^2 + (v_{x_2} + \frac{1}{2} w_{x_2}^2)^2 \right)
\]

\[
+ \frac{1 - \nu}{2} \left( u_{x_2} + v_{x_1} + w_{x_1} w_{x_2} \right)^2
\]

\[
+ \frac{h^3}{12} \left( w_{x_1}^2 w_{x_1}^2 + w_{x_2}^2 w_{x_2}^2 + w_{x_1}^2 w_{x_2}^2 \right) dx_2 dx_1
\]
The potential energy of the external applied forces will be defined as the negative of the work done by fluid forces acting on the plate, where $K$ is a damping constant.

\[ A = u(f_1 - Ku_t) + v(f_2 - Kv_t) + w(f_3 - Kw_t) \]
According to Hamilton’s principle, the progression of all physical systems minimizes the time integral of the Lagrangian, which is to say the variation of the Lagrangian will always be zero, i.e.

$$\delta \int_{t_0}^{t_1} [(T - U) + A] dt = 0$$

Expanding this integral, we obtain the governing equations.
\[ 0 = \delta \int_{t_0}^{t_1} \int_0^a \int_0^a \frac{\rho h}{2} (u_t^2 + v_t^2 + w_t^2) \]
\[ - \frac{Yh}{(1 - \nu^2)} ((u_{x_1} + \frac{1}{2} w_{x_1})^2 + (v_{x_2} + \frac{1}{2} w_{x_2})^2) \]
\[ + \frac{1 - \nu}{2} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})^2) \]
\[ + \frac{Yh^3}{12(1 - \nu^2)} (w_{x_1} w_{x_1}^2 + w_{x_2} w_{x_2} + w_{x_1} w_{x_2})^2 \]
\[ + u(f_1 - Ku_t) + v(f_2 - Kv_t) + w(f_3 - Kw_t) dx_2 dx_1 dt \]
Using integration by parts to handle each term, the variation and the first spacial and temporal derivatives of the variation are zero at the limits of integration, therefore each boundary term is cancelled. After collecting all of the terms with contain $\delta u$, $\delta v$, $\delta w$, we can separate the integral into three parts as follows:
Calculus of Variation - $\delta u$ component

$$0 = \int_{t_0}^{t_1} \int_0^a \int_0^a \delta u (-\rho h u_{tt} + \frac{Yh}{(1 - \nu^2)} [u_{x_1} + \frac{1}{2} w_{x_1}^2]_{x_1}$$

$$+ \frac{1 - \nu}{2} [u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} + Ku_t + f_1) dx_2 dx_1 dt$$
\[ 0 = \int_{t_0}^{t_1} \int_0^a \int_0^a \delta v (-\rho hv_{tt} + \frac{Yh}{(1 - \nu^2)}([v_{x_2} + \frac{1}{2}w_{x_2}^{2}]_{x_2} \\
+ \frac{1 - \nu}{2} [u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_1}) + Kv_t + f_2) \, dx_2 \, dx_1 \, dt \]
Calculus of Variation - \( \delta w \) component

\[
0 = \int_{t_0}^{t_1} \int_{0}^{a} \int_{0}^{a} \delta w \left( -\rho hw_{tt} + \frac{Yh}{(1-\nu^2)} \left( [w_{x_1}(u_{x_1} + \frac{1}{2}w_{x_1}^2)]_{x_1} \right. \right.
+ \left. \left. [w_{x_2}(v_{x_2} + \frac{1}{2}w_{x_2}^2)]_{x_2} + \frac{1-\nu}{2} [w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} \right. \right.
+ \left. \left. [w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1} \right) \right.
+ \frac{Yh^3}{12(1-\nu^2)} (w_{x_1x_1x_1x_1} + w_{x_2x_2x_2x_2} + 2w_{x_1x_1x_2x_2})
+ \left. Kw_t + f_3 \right) dx_2 dx_1 dt
\]
Final Model

\[ f_1 = u_{tt} + Cu_t - D_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} - E[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} \]
\[ f_2 = v_{tt} + Cv_t - D_1[v_{x_2} + \frac{1}{2}(w_{x_2})^2]_{x_2} - E[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_1} \]
\[ f_3 = w_{tt} + Cw_t + D[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \]
\[ -D_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - D_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \]
\[ -E[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - E[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1} \]

where \( C, D, E, \) and \( D_1 \) are all constants depending on the system.
Part 2: Analytic Stability

We will show for any transversal force $f_3$ the energy of the system changes proportionally to the force. In other words, our choice of initial conditions won’t cause the system to experience *flutter* or other disastrous instabilities.
Multiply the first equation by $u_t$ and $aCu$, the second by $v_t$ and $aCv$, and the third by $w_t$ and $aCw$; where $0 \leq a \leq 1$.

After this, add the resulting equations. For example, the first equation becomes:

\[
\frac{1}{2} [u_t]^2_t + Cu_t^2 + aCu_{tt}u + \frac{a}{2} C^2 [u^2]_t = \\
D_1[u_{x_1} + \frac{1}{2} w_{x_1}^2]_{x_1} u_t + aD_1 Cu_{x_1} + \frac{1}{2} w_{x_1}^2]_{x_1} u + \\
E[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} u_t + aEC[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} u + \\
f_1(u_t + aCu)
\]
The overall goal of what we are doing is to reduce our now *more* complicated system into a (relatively) simple ordinary differential equation. Using integration by parts and some algebra tricks, this is possible, though it takes a lot of work. If you want the full derivation, email me.
Ordinary Differential Equation

\[
[M_1]_t + M_2 = \int_0^L \int_0^L f_3(w_t + \frac{aCW}{2})dx_1dx_2
\]

\[
M_1 = \int_0^L \int_0^L \frac{a}{2}(u_t + Cu)^2 + \frac{a}{2}(v_t + Cv)^2 + \frac{a}{4}(w_t + Cw)^2 + \frac{1-a}{2}u_t^2 + \frac{1-a}{2}v_t^2
\]

\[
+ \left(\frac{1}{2} - \frac{a}{4}\right) w_t^2 + \frac{D_1}{2} \left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + \frac{D_1}{2} \left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2
\]

\[
+ \frac{E}{2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2 + \frac{D}{2}(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2)dx_1dx_2
\]

\[
M_2 = \int_0^L \int_0^L (1-a)C_u^2 + (1-a)C_v^2 + (1-\frac{a}{2})Cw_t^2
\]

\[
+ aCD_1 \left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + aCD_1 \left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2
\]

\[
+ aEC(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2 + \frac{aCD}{2}(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2)dx_1dx_2
\]
Bounding the Equation

\[ [M_1]_t + M_2 = \int_0^L \int_0^L f_3(w_t + \frac{aCw}{2}) dx_1 dx_2 \]

One thing to notice is this equation looks a lot like,

\[ \frac{d}{dt} [y(t)] + x(t) = f(t) \]

And if we can get that to look like,

\[ \frac{d}{dt} [y(t)] + y(t) \leq f(t) \]

\[ \frac{d}{dt} [e^t y(t)] \leq f(t)e^t \]

\[ e^T y(T) - y(0) \leq \int_0^T f(t)e^t dt \]

\[ y(T) \leq y(0)e^{-T} + e^{-T} \int_0^T f(t)e^t dt \]
What To Do With $M_2$

\[ [M_1]_t + M_2 = \int_0^L \int_0^L f_3(w_t + \frac{aCw}{2}) \, dx_1 \, dx_2 \]

It would be ideal to find a function $M_2^*$ so that $M_2^*$ is bounded above by some factor of $M_1$ because that would provide us with the inequality we seek. Thus we begin our search for the elusive $M_2^*$ by repeatedly throwing inequalities at our differential equation.

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Mathematical Modeling of Large Deformations on a Non-Linear Plate
Using Young’s Inequality and the Poincaré Inequality we get,

\[
[M_1]_t + C \int_0^L \int_0^L (1 - a)u_t^2 + (1 - a)v_t^2 + \frac{1}{2}(1 - \frac{a}{2})w_t^2
\]

\[+ aD_1 \left( u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right)^2 + aD_1 \left( v_{x_2} + \frac{1}{2}(w_{x_2})^2 \right)^2
\]

\[+ aE(u_{x_2} + v_{x_1} + w_{x_1x_2})^2 + \frac{aD}{4}(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2) \, dx_1 \, dx_2
\]

\[\leq \int_0^L \int_0^L f_3(w_t + \frac{aCw}{2}) \, dx_1 \, dx_2
\]

or setting \( M_2^* \) to be the terms in red,

\[
[M_1]_t + CM_2^* \leq f_3^2 \left( \frac{aC}{8\varepsilon_1} + \frac{1}{4\varepsilon_2} \right) \, dx_1 \, dx_2
\]

Goal: We will use \( M_2^* \) as the energy function to bound changes in the solutions in order to prove our stability result.
By Construction

\[ M_2^* \leq 2M_1 \]

\[ M_1 = \int_0^L \! \int_0^L \frac{a}{2} (u_t + Cu)^2 + \frac{a}{2} (v_t + Cv)^2 + \frac{a}{4} (w_t + Cw)^2 + \frac{1 - a}{2} u_t^2 + \frac{1 - a}{2} v_t^2 \]

\[ + \left( \frac{1}{2} - \frac{a}{4} \right) w_t^2 + \frac{D_1}{2} \left( u_{x_1} + \frac{1}{2} (w_{x_1})^2 \right)^2 + \frac{D_1}{2} \left( v_{x_2} + \frac{1}{2} (w_{x_2})^2 \right)^2 \]

\[ + \frac{E}{2} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})^2 + \frac{D}{2} (w_{x_1 x_1}^2 + 2w_{x_1 x_2}^2 + w_{x_2 x_2}^2) \, dx_1 \, dx_2 \]

\[ M_2^* = \int_0^L \! \int_0^L (1 - a) u_t^2 + (1 - a) v_t^2 + \frac{1}{2} (1 - a) w_t^2 \]

\[ + aD_1 \left( u_{x_1} + \frac{1}{2} (w_{x_1})^2 \right)^2 + aD_1 \left( v_{x_2} + \frac{1}{2} (w_{x_2})^2 \right)^2 \]

\[ + aE (u_{x_2} + v_{x_1} + w_{x_1 x_2})^2 + \frac{aD}{4} (w_{x_1 x_1}^2 + 2w_{x_1 x_2}^2 + w_{x_2 x_2}^2) \, dx_1 \, dx_2 \]
First Order Linear Differential Equation

\[ [M_1]_t + M_2 = \int_0^L \int_0^L f_3(w_t + \frac{aCw}{2}) d\tau \]

\[ [M_1]_t + CM_2^* \leq \int_0^L \int_0^L f_3^2 \left( \frac{aC}{8\varepsilon_1} + \frac{1}{4\varepsilon_2} \right) d\tau \]

\[ \frac{1}{2} M_2^*(t) - M_1(0) + C \int_0^t M_2^*(\tau) d\tau \leq \left( \frac{aC}{8\varepsilon_1} + \frac{1}{4\varepsilon_2} \right) \int_0^t \| f_3 \|_{L_2}^2 d\tau \]

\[ M_2^*(t) + 2C \int_0^t M_2^*(\tau) d\tau \leq 2M_1(0) + \left( \frac{aC}{4\varepsilon_1} + \frac{1}{2\varepsilon_2} \right) \int_0^t \| f_3 \|_{L_2}^2 d\tau \]

\[ \frac{d}{dt} \left[ e^{2Ct} \int_0^t M_2^*(\tau) d\tau \right] \leq 2M_1(0)e^{2Ct} + e^{2Ct} \left( \frac{aC}{4\varepsilon_1} + \frac{1}{2\varepsilon_2} \right) \int_0^t \| f_3 \|_{L_2}^2 d\tau \]
Our Theorem

For any non-linear plate and any transversal force, $f_3$, the energy measure, $M_2^*$, satisfies the following inequality for any given time.

$$
\int_0^T M_2^*(\tau) d\tau \leq \frac{M_1(0)}{C} \left[ 1 - e^{-2CT} \right] + e^{-2CT} \left[ \frac{aC}{4\varepsilon_1} + \frac{1}{2\varepsilon_2} \right] \int_0^T e^{2Ct} \left( \int_0^t ||f_3||_{L_2}^2 d\tau \right) dt
$$

where

$$
M_2^* = \int_0^L \int_0^L (1 - a)u_t^2 + (1 - a)v_t^2 + \frac{1}{2}(1 - \frac{a}{2})w_t^2
$$

$$
+ aD_1 \left( u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right)^2 + aD_1 \left( v_{x_2} + \frac{1}{2}(w_{x_2})^2 \right)^2
$$

$$
+ aE(u_{x_2} + v_{x_1} + w_{x_1x_2})^2 + \frac{aD}{4}(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2) \, dx_1 dx_2
$$

where $u, v, w$ are axial displacements and $w$ is transverse displacement, $\varepsilon_1, \varepsilon_2, C, M_1(0)$ are real numbers dependent on the system.
Part 3: Numerical Stability

The next step, now that we have proven the system to be analytically stable, is to show numerical stability and use parameter identification studies to validate our model using an explicit FTCS finite difference method. At the moment, we are working on proofs of stability similar to the classic "Von Neumann stability" to get convergence results.
Future Work

- Continue numerical validation using finite difference method and finite element method
- Error convergence
- Nonlinear material constitutive law
- Approximation of other nonlinearities
- Parameter identification studies to validate the model
- Allow $f_1$ and $f_2$ to be nontrivial
- Allow $\int_0^t ||f_3||^2_{L_2} d\tau$ to be bounded or constant
- Couple the structural model to a fluid model


For further questions on this presentation, contact us at the following email addresses:

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