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Effect of magnetic and nonmagnetic impurities on highly anisotropic superconductivity

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We generalize Abrikosov-Gor'kov solution of the problem of weakly coupled superconductors with impurities to the case of a multiband superconductor with arbitrary interband order parameter anisotropy, including interband sign reversal of the order parameter. The solution is given in terms of the effective (renormalized) coupling matrix and describes not only T_c suppression but also renormalization of the superconducting gap basically at all temperatures. In many limiting cases we find analytical solutions for the critical temperature suppression. We illustrate our results by numerical calculations for two-band model systems.
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I. INTRODUCTION

Recent advances in the field of high-temperature superconductivity, in particular the discovery of the strong anisotropy of the order parameter, have stimulated interest in the old problem of the effect of (magnetic and nonmagnetic) impurity scattering on superconductivity with high anisotropy. In a number of theoretical papers published within the last few years qualitatively new phenomena were uncovered.[1–6] Moreover, detailed experimental studies of the effect of impurities in high-temperature superconductors are underway (see, e.g., Refs. 7–10 and references therein).

A specific, but representative case of anisotropic superconductivity is multiband superconductivity (e.g., Refs. 2 and 11–15), where the order parameter is different in different bands. Allen showed in 1978 (Ref. 16) (see also Ref. 17) that a superconductor with a general anisotropy can be treated within the same mathematical formalism as a multiband superconductor, if one expands the order parameter, pairing interaction, and impurity scattering in terms of the Fermi surface harmonics. In this paper we derive a general formula, analogous to the Abrikosov-Gor'kov formula for isotropic superconductors,[18] but valid for an arbitrary multiband system. According to Allen's formalism, this result is easily generalizable to superconductivity with arbitrary angular anisotropy. We will also show explicit results for various limiting cases to illustrate the physics of the interplay between impurity scattering and gap function anisotropy. We will illustrate the results on a model system with strong interband anisotropy, namely, one where superconductivity in one of two bands is induced by the interband proximity effect. We will use the Born scattering limit for the impurity scattering cross section, since this approximation captures correctly the effect of impurities and the relation to the existing literature is most transparent.

II. GENERAL THEORY

Following the standard way of including impurity scattering in BCS theory in Born approximation,[18] one writes the equations for the renormalized frequency $\widetilde{\omega}_n$ and order parameter $\widetilde{\Delta}_n$ (n is the Matsubara index), which completely define the superconductive properties of the system:

$$\hbar\widetilde{\omega}_{\alpha n}=\hbar\omega_n+\sum_\beta \frac{\hbar^2\widetilde{\omega}_{\beta n}}{2Q_{\beta n}}(\gamma_{\alpha\beta}+\gamma^s_{\alpha\beta}), \tag{1a}$$

$$\widetilde{\Delta}_{\alpha n}=\Delta_\alpha+\sum_\beta \frac{\hbar^2\Delta_{\beta n}}{2Q_{\beta n}}(\gamma_{\alpha\beta}-\gamma^s_{\alpha\beta}), \tag{1b}$$

$$\Delta_\alpha=2\pi T\sum_{\beta,n}^{0<\omega_n<\omega_D}\Lambda_{\alpha\beta}\widetilde{\Delta}_{\beta n}/Q_{\beta n}. \tag{1c}$$

The general form of these equations for strong coupling and general anisotropy in terms of the Fermi surface harmonics can be found in Ref. 17. Note that according to Allen's terminology we work in the disjoint representation, where Fermi surface harmonics are defined separately for each sheet of the Fermi surface, and take into account only the lowest harmonic for each sheet. Other notations in Eqs. (1) have their usual meaning: $\omega_n=(2n+1)\pi T$, $Q_{\alpha n}=\sqrt{\widetilde{\omega}^2_{\alpha n}+\widetilde{\Delta}^2_{\alpha n}}$, $\gamma_{\alpha\beta}=U_{\alpha\beta}N_\beta$ is the scattering rate matrix due to nonmagnetic impurities, and $\gamma^s_{\alpha\beta}=U^s_{\alpha\beta}N_\beta$ is the same for magnetic impurities. The coupling matrix Λ is defined in the same way as Allen's matrix $\lambda_{\alpha\alpha'}$,[16] $\Lambda_{\alpha\alpha'}=V^{\text{pairing}}_{\alpha\alpha'}N_{\alpha'}$. Here N_α is the partial density of states at the Fermi level in the band α. The scattering potential U and the pairing potential V^{pairing} are symmetric matrices, while γ, γ^s, and Λ are not. We shall also introduce the following useful notations:

$$\lambda_\alpha=\sum_\beta\Lambda_{\alpha\beta},\quad \lambda=\sum_\alpha\lambda_\alpha N_\alpha/N,\quad N=\sum_\alpha N_\alpha, \tag{2}$$

where N is the total density of states, λ_α are partial electron-phonon coupling constants, which define the electron mass renormalization in the band α, and λ is the total isotropic coupling constant, which enters the BCS and Eliashberg equations for isotropic constant gap superconductivity. Analogously, we shall introduce the partial scattering rates

$$G_\alpha^\pm=\sum_\beta \Gamma_{\alpha\beta}^\pm,\quad \Gamma_{\alpha\beta}^\pm=\gamma_{\alpha\beta}\pm\gamma_{\alpha\beta}^s,\quad g_{\alpha\beta}^\pm=\delta_{\alpha\beta}G_\alpha^\pm-\Gamma_{\alpha\beta}^\mp. \tag{3}$$

At temperatures close to T_c one can linearize Eqs. (1) with respect to Δ. To do so, we introduce, as usual, the renormalization function Z and the gap function Δ':

$$Z_{\alpha n}=\widetilde{\omega}_{\alpha n}/\omega_n,\quad \Delta'_{\alpha n}=\widetilde{\Delta}_{\alpha n}/Z_{\alpha n},\quad Z_{\alpha n}=1+G_\alpha^+/2\omega_n,$$

$$\Delta'_{\alpha n}(1+G_\alpha^+/2\omega_n)=\Delta_\alpha+\sum_{\alpha'}\Delta'_{\alpha' n}\Gamma_{\alpha\alpha'}^-/2\omega_n, \tag{4}$$

which we can solve for Δ':

$$\Delta'_{\alpha n}=\sum_{\alpha'}\Delta_{\alpha'}(\delta_{\alpha\alpha'}+g_{\alpha\alpha'}^+/2\omega_n)^{-1}. \tag{5}$$

Now from Eq. (1c) it follows that

$$\Delta_\alpha=2\pi T\sum_{\beta n}^{0<\omega_n<\omega_D}\Lambda_{\alpha\beta}\omega_n^{-1}\sum_\gamma\Delta_\gamma\left(\delta_{\gamma\beta}+\frac{g_{\gamma\beta}^+}{2\omega_n}\right)^{-1}. \tag{1d}$$

For weak ($\gamma\ll 2\pi T_c$) and for intermediate ($\gamma\ll\omega_D$) scattering the usual trick with subtracting the clean limit $g=0$ can be applied, and extending summation to infinity [a useful matrix formula is $(\hat{1}+\hat{A})^{-1}=\hat{1}-\hat{A}(\hat{1}+\hat{A})^{-1}$], one gets

$$\Delta_\alpha=\sum_{\beta\gamma}\Lambda_{\alpha\beta}[L\delta_{\beta\gamma}-X_{\beta\gamma}]\Delta_\gamma,$$

$$X_{\alpha\beta}=2\pi T_c\sum_n\sum_\gamma(g_{\gamma\beta}^+/2)\omega_n^{-1}(\omega_n\delta_{\alpha\gamma}+g_{\alpha\gamma}^+/2)^{-1}, \tag{6}$$

where $L=\ln(2\gamma^*\omega_D/\pi T_c)$. $\gamma^*\approx 1.78$ is the Euler constant. By introducing the eigensystem of g^+, $g_{\alpha\beta}^+=\Sigma_\gamma R_{\alpha\gamma}^{-1}d_\gamma R_{\gamma\beta}$, we can express X in terms of the difference between the two incomplete gamma functions [$\psi(x)\equiv\Sigma_{n\geqslant 0}(n+x)^{-1}$]:

$$X_{\alpha\beta}=2\pi T_c\sum_\gamma R_{\alpha\gamma}^{-1}\sum_n\omega_n^{-1}(d_\gamma/2)(\omega_n+d_\gamma/2)^{-1}R_{\gamma\beta}$$

$$=\sum_\gamma R_{\alpha\gamma}^{-1}\chi(d_\gamma/4\pi T_c)R_{\gamma\beta}, \tag{7}$$

with $\chi(x)=\psi(1/2)-\psi(1/2+x)$, which is the standard definition of the matrix function $\hat{X}=\chi(\hat{g}^+/4\pi T_c)$.

This result is analogous to the classic one of Abrikosov and Gor'kov[18] (AG), but includes arbitrary anisotropy. Now solving Eqs. (6) for L, we find

$$\Delta_\alpha=\sum_\gamma(\Lambda_{\alpha\gamma}^{-1}+X_{\alpha\gamma})^{-1}L\Delta_\gamma, \tag{8}$$

which means that now T_c is defined by the *effective* matrix $\Lambda_{\rm eff}=(\Lambda^{-1}+X)^{-1}$. As is well known in multiband superconductivity theory,[19] in this case T_c is defined by the usual BCS equation $T_c=(2\gamma^*\omega_D/\pi)\exp(-1/\lambda_{\max})$, where $\lambda_{\max}$ is the maximal eigenvalue of the matrix Λ (in our case, of the matrix $\Lambda_{\rm eff}$). As can be seen immediately from Eqs. (6)–(8) and the definition of $g_{\alpha\beta}$, diagonal nonmagnetic scattering rates $\gamma_{\alpha\beta}$ have dropped out from Eq. (8). This is the manifestation of the Anderson theorem for a many band case: intraband scattering does not influence T_c (in the considered Born limit). As will be discussed below, this argument works only for intraband nonmagnetic scattering, while all others are, in principle, pair breaking.

Up to the second order in Λ (assuming that ΛX is small),

$$\Lambda_{\rm eff}=\Lambda-\Lambda X\Lambda. \tag{9}$$

If we recall that Δ forms the eigenvector of Λ corresponding to its maximal eigenvalue $\lambda_{\rm eff}$, we can immediately write the lowest-order correction to $\lambda_{\rm eff}$:

$$\delta\lambda_{\rm eff}=-\lambda_{\rm eff}^2\sum_{\alpha\beta}\Delta_\alpha X_{\alpha\beta}\Delta_\beta/\sum_\alpha\Delta_\alpha^2. \tag{10}$$

In the strong scattering case (here and below ''strong scattering'' means a scattering rate which is strong in the superconducting energy scale, $\gamma\gg\omega_D$, but not as strong as to violate the Born approximation) this formalism cannot be used. Instead, one should use Eq. (1d) directly.

III. CRITICAL TEMPERATURE

A. Weak scattering

Let us consider explicitly some interesting limiting cases. For weak scattering ($\gamma_{\alpha\beta},\gamma_{\alpha\beta}^s\ll T_c$) one can use Eq. (10), and expand $\chi(x\to 0)=\pi^2x/2$ and write

$$\delta T_c/T_c=\delta\lambda_{\rm eff}/\lambda_{\rm eff}^2=-\frac{\Sigma_{\alpha\beta}\Delta_\alpha X_{\alpha\beta}\Delta_\beta}{\Sigma_\alpha\Delta_\alpha^2}$$

$$\approx-\frac{\pi\Sigma_{\alpha\beta}\Delta_\alpha g_{\alpha\beta}^+\Delta_\beta}{8T_c\Sigma_\alpha\Delta_\alpha^2}. \tag{11}$$

When all Δ's are equal (isotropic case), the standard Abrikosov-Gor'kov (AG) result is recovered: $\delta T_c/T_c=-\pi\Sigma_{\alpha\beta}(\Gamma_{\alpha\beta}^+-\Gamma_{\alpha\beta}^-)/8T_{c0}=-(\pi/4T_{c0})\Sigma\gamma^s$; that is, nonmagnetic scattering falls out. On the other hand, in the anisotropic case only the intraband nonmagnetic scattering falls out of Eq. (11), as, for instance, in the two-band case:

$$\delta T_c/T_c=-\frac{\pi}{8T_c}\hat{\Delta}\cdot\begin{pmatrix}2\gamma_{11}^s+\gamma_{12}^s+\gamma_{12} & \gamma_{12}^s-\gamma_{12}\\ \gamma_{21}^s-\gamma_{21} & 2\gamma_{22}^s+\gamma_{21}^s+\gamma_{21}\end{pmatrix}\cdot\hat{\Delta}/\hat{\Delta}\cdot\hat{\Delta}$$

$$=-\frac{\pi[\Delta_1^2(2\gamma_{11}^s+\gamma_{12}^s+\gamma_{12})+\Delta_1\Delta_2(\gamma_{12}^s+\gamma_{21}^s-\gamma_{12}-\gamma_{21})+\Delta_2^2(2\gamma_{22}^s+\gamma_{21}^s+\gamma_{21})]}{8T_c(\Delta_1^2+\Delta_2^2)}. \tag{12}$$

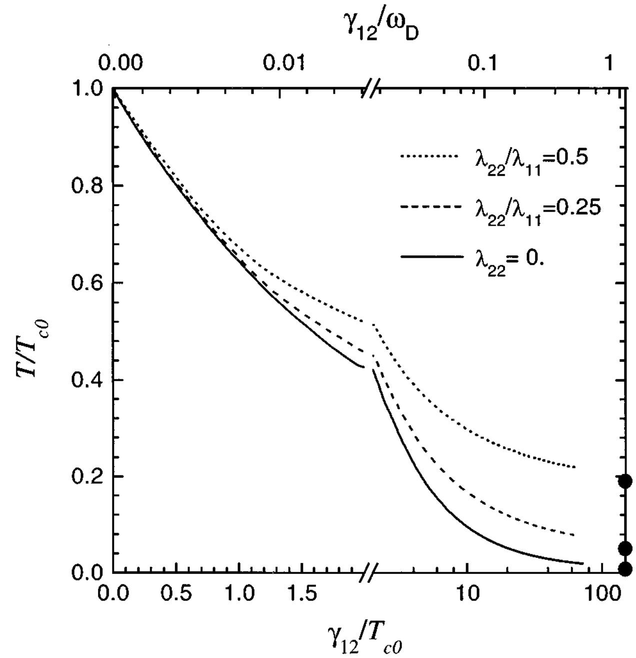

FIG. 1. T_c suppression by interband scattering. Dots on the right axis show the asymptotic value of T_c at $\gamma\rightarrow\infty$, according to Eq. (22).

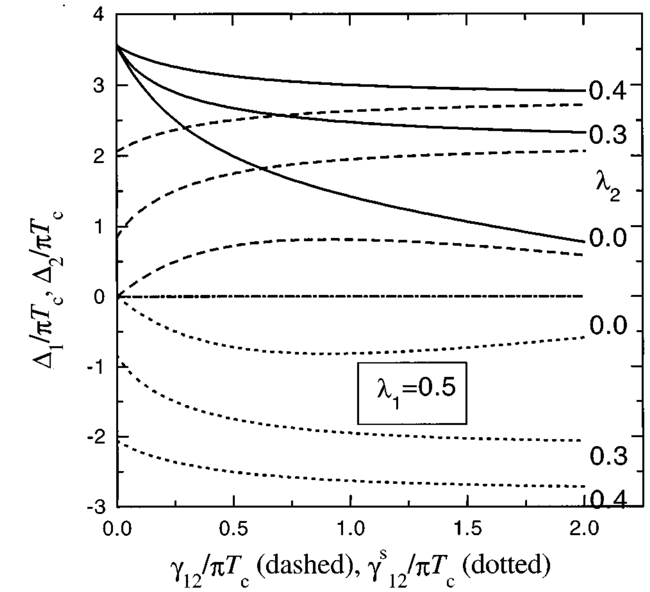

FIG. 2. Dependence of the order parameters $\Delta_{1,n=0}$ and $\Delta_{2,n=0}$ in two bands in a two-band model with $\lambda_{12}=\lambda_{21}=0$ and different ratios $\lambda_{22}/\lambda_{11}$, on the interband impurity scattering. Solid lines show the order parameter in the first (''superconducting''); dashed and dotted lines show the order parameter in the second band, where superconductivity is induced by impurity scattering. Dashed lines correspond to nonmagnetic interband scattering, dotted lines to magnetic interband scattering.

The main point of the AG theory[18] is that γ^s enters equations for ω and Δ with opposite signs. That is why the magnetic impurities appear to be pair breakers and the nonmagnetic ones not. The above solution shows explicitly that in the multiband case of Eqs. (1) only intraband nonmagnetic scattering does not influence T_c ($\gamma_{\alpha\beta}$ drops out). In an interesting limit of two bands, in which one band is superconducting and another is not, $\lambda_{11}\neq 0$, $\lambda_{12}=\lambda_{21}=\lambda_{22}=0$, it follows from Eq. (12) that

$$\delta T_c/T_c=-\frac{\pi}{8T_c}(2\gamma_{11}^s+\gamma_{12}^s+\gamma_{12}), \tag{13}$$

where the first term is the usual AG T_c suppression, and the last two show that the pair-breaking influence of the nonsuperconducting band is the same both for magnetic and nonmagnetic scattering. However, the sign of the order parameter, induced in the second band, is different: the same for nonmagnetic and the opposite for magnetic scattering (cf. the $\lambda_2=0$ curves in Fig. 2). Such a sign reversal is discussed in more detail later in the paper.

In the next order in $\lambda_{\beta,\alpha\neq 1}$ the additional correction to $\delta T_c/T_c$ is $(\delta T_c/T_c)_1=(-\pi/8T_c)[(\gamma_{21}^s-\gamma_{21})\lambda_{12}+(\gamma_{12}^s-\gamma_{12})\lambda_{21}]/(\lambda_{11}-\lambda_{22})$ (this corresponds to the so-called interband tunneling, specific cases of which are considered in the literature[15]). In the limit of $\gamma_{ij}^s=0$ the above expression coincides with that derived in Ref. 15. Since $\gamma_{12}/\gamma_{21}=\lambda_{12}/\lambda_{21}=N_2/N_1$, the last expression can also be written as

$$(\delta T_c/T_c)_1=-\frac{\pi}{4T_c}(\gamma_{12}^s-\gamma_{12})\lambda_{21}/(\lambda_{11}-\lambda_{22}). \tag{14}$$

Note that if $\lambda_{21}=0$, suppression of T_c is independent of λ_{22}, as long as $\lambda_{11}>\lambda_{22}$. It is clearly seen, for instance, in the left-hand part of Fig. 1, where the suppression rate for $\lambda_{21}=0$ and various λ_{22} is shown, and is practically independent of λ_{22}. For producing this figure we have solved Eqs. (1) numerically for two bands, assuming $\lambda_{21}=\lambda_{12}=0$, $\gamma_{\alpha\beta}^s=\gamma_{11}=\gamma_{22}=0$, and $\gamma_{12}=\gamma_{21}$. In full agreement with Eqs. (13) and (14), T_c is first suppressed linearly with the rate $\pi\gamma_{12}/8T_{c0}$; then, at $\gamma_{12}\sim T_c$ it starts to deviate from linearity and, as will be proved later in the paper, saturates at some value depending on λ_{22}.

Another important limiting case, also often considered in the literature, is the limit of weak anisotropy. Let us assume that $\Delta_\alpha=\Delta+\delta\overline{\Delta}_\alpha$, where $|\delta\Delta_\alpha|\ll\overline{\Delta}$. The pair-breaking effect of magnetic impurities is then given by isotropic AG theory, and so it is sufficient to consider only nonmagnetic scattering. Let us also take, for simplicity, an isotropic scattering $g_{\alpha\beta}^+=\gamma(\delta_{\alpha\beta}-1)$. Then Eq. (11) gives

$$\delta T_c/T_c=-\frac{\pi\gamma_{\text{tot}}}{8T_c}\frac{(\overline{\Delta^2}-\overline{\Delta}^2)}{\overline{\Delta}^2}\approx-\frac{\pi\gamma_{\text{tot}}}{8T_c}\frac{\overline{\delta\Delta^2}}{\overline{\Delta}^2}, \tag{15}$$

where γ_{tot} is the total nonmagnetic scattering, summed over all bands (or Fermi harmonics). Thus in the case of weak anisotropy the T_c suppression is given by the AG formula with an effective scattering rate $\gamma_{\text{eff}}^s=\gamma^s+(\overline{\delta\Delta^2}/\overline{\Delta}^2)\gamma$. This result has often been obtained for angular gap anisotropy.[20,21,3]

B. Interband sign reversal of the order parameter

Returning to the two-band case, we observe that Eq. (8) is invariant with respect to a simultaneous change of signs of λ_{12} and λ_{21} and an interchange of nondiagonal magnetic and nonmagnetic scattering $\gamma_{12}\to\gamma^s_{12}$, $\gamma_{21}\to\gamma^s_{21}$. This remarkable property is the consequence of the symmetry of the matrix $\hat{\Lambda}_{\rm eff}$ with respect to the above transformations with a simultaneous reversal of the relative signs of the order parameters Δ_1,Δ_2. One manifestation of this phenomenon is discussed above for induced superconductivity. Another illustration is given by the symmetric case $\lambda_{11}=\lambda_{22}\equiv\lambda_\parallel$ and $\lambda_{12}=\lambda_{21}\equiv\lambda_\perp$. Then it follows from Eq. (8) that with a sign change of $\lambda_\perp$ the role of magnetic and nonmagnetic interband scattering is completely reversed. Namely, for positive $\lambda_\perp$ (Δ_1 and Δ_2 have the same signs) only magnetic interband scattering suppresses T_c according to $(T_{c0}-T_c)/T_c\approx\pi(\gamma^s_{12}+\gamma^s_{21})/8T_c$. In the opposite case of negative $\lambda_\perp$ (Δ_1 and Δ_2 have different signs) the magnetic impurities do not influence T_c but the nonmagnetic ones suppress it according to $(T_{c0}-T_c)/T_c\approx\pi(\gamma_{12}+\gamma_{21})/8T_c$. The case of arbitrary $\lambda_{22}/\lambda_{11}$ is shown in Fig. 2, where we show the numerical solution of Eqs. (1) for the same model as we used in Fig. 1: $\lambda_{11}=0.5$ is fixed, and λ_{22} changes from 0 to 0.4. Both magnetic ($\gamma^s_{12}\neq0$, $\gamma_{12}=0$) and nonmagnetic ($\gamma^s_{12}=0$, $\gamma_{12}\neq0$) impurities are considered. In the first band the order parameter is suppressed equally by magnetic and nonmagnetic impurities: The solid curves in Fig. 2 are the same for both cases. The order parameter in the second band has the same absolute value for pure magnetic or for pure nonmagnetic scattering, but its sign is different in the two cases. Moreover, even if $\lambda_{12},\lambda_{21}\neq0$, but $\lambda_{12},\lambda_{21}\ll\lambda_{11},\lambda_{22}$, there still is a possibility of the interband sign reversal of the gap due to magnetic impurities. This happens when nondiagonal elements in the effective Λ matrix in Eq. (8) become negative, $\lambda^{\rm eff}_{12}=\lambda_{12}+\pi\lambda_{11}\lambda_{22}(\gamma_{12}-\gamma^s_{12})/8T_{c0}$, which does happen if γ^s_{12} is sufficiently large. Then the order parameters in different bands have different signs; i.e., a solution with $\mathrm{sgn}(\Delta_\beta)=-\mathrm{sgn}(\Delta_\alpha)$ corresponds to a minimum energy. This sign reversal leads to an interesting effect: If one starts from a pure superconductor with weak interband coupling and suppresses T_c by adding interband magnetic impurity scattering, at some critical scattering strength the suppression rate drops drastically. The final comment to Fig. 2 is that it shows either solely magnetic or solely nonmagnetic scattering. When both kinds of scattering are present, the order parameter in the second band is much smaller than in either pure case and becomes zero when magnetic and nonmagnetic scatterings are equally strong. A numerical illustration of this effect can be found in Ref. 2.

This situation is closely analogous to the known case of d pairing, where isotropic nonmagnetic impurity scattering leads to an AG T_c suppression, but with a factor of 2 smaller coefficient (cf. Refs. 22 and 23). If we label those parts of the Fermi surface that have positive order parameter as 1 and those which have negative order parameter as 2, then only in the "interband channel" are the nonmagnetic impurities pair breaking, while the magnetic impurities are pair breaking only in the "intraband channel." Correspondingly, the effective pair-breaking scattering rate will be $\gamma^s_{11}+\gamma^s_{22}+\gamma_{12}+\gamma_{21}=(\gamma_{\rm tot}+\gamma^s_{\rm tot})/2$. Note that isotropic magnetic scattering results in exactly as much pair breaking in terms of T_c as isotropic nonmagnetic scattering, contrary to the popular misconception that only nonmagnetic impurities are suppressing T_c in d-wave superconductors. Interestingly, if isotropic magnetic and nonmagnetic scatterings are both present and have equal strength, the T_c suppression rate is the same for the s- and d-wave superconductors. If only magnetic scattering is present, T_c is suppressed twice faster in an s superconductor. In fact, most of these statements are not specific for d pairing, but are true for any superconductor with zero average order parameter and nonzero average square for the order parameter. Let us, for example, prove that in such a superconductor isotropic magnetic and nonmagnetic scatterings both have the same effect on T_c. According to Eq. (11), T_c suppression rate is proportional to

$$\begin{aligned}\langle\Delta_{\mathbf k}g^+_{\mathbf{kk}'}\Delta_{\mathbf k}\rangle_{\mathbf{k},\mathbf{k}'}&=\langle\Delta^2_{\mathbf k}G^+_{\mathbf k}\rangle_{\mathbf k}-\langle\Delta_{\mathbf k}\Gamma^-_{\mathbf{kk}'}\Delta_{\mathbf k'}\rangle_{\mathbf{kk}'}\\&=\langle\Delta^2_{\mathbf k}\langle\Gamma^+_{\mathbf{kk}'}\rangle_{\mathbf k'}\rangle_{\mathbf k}-\langle\Delta_{\mathbf k}\Gamma^-_{\mathbf{kk}'}\Delta_{\mathbf k'}\rangle_{\mathbf{kk}'},\end{aligned}\tag{16}$$

where we used $\mathbf k$ and $\mathbf k'$ for indices to emphasize that the formalism is valid both for interband or for angular anisotropy. For isotropic scattering $\gamma_{\mathbf{k,k}'}=\gamma$, $\gamma^s_{\mathbf{k,k}'}=\gamma^s$, this equation reduces to

$$\begin{aligned}&\langle\Delta^2_{\mathbf k}\rangle(\gamma+\gamma^s)-\langle\Delta_{\mathbf k}\Delta_{\mathbf k'}\rangle(\gamma-\gamma^s)\\&\quad=(\langle\Delta^2_{\mathbf k}\rangle+\langle\Delta_{\mathbf k}\Delta_{\mathbf k'}\rangle)\gamma^s+(\langle\Delta^2_{\mathbf k}\rangle-\langle\Delta_{\mathbf k}\Delta_{\mathbf k'}\rangle)\gamma.\end{aligned}\tag{17}$$

For isotropic s-wave superconductors, $\langle\Delta^2_{\mathbf k}\rangle=\langle\Delta_{\mathbf k}\Delta_{\mathbf k'}\rangle=\Delta^2$, and the T_c suppression rate does not depend on γ. For a superconductor where $\langle\Delta_{\mathbf k}\Delta_{\mathbf k'}\rangle=0$, $\langle\Delta^2_{\mathbf k}\rangle\neq0$, a specific case of which is a d-wave superconductor, the suppression rate is proportional to $(\gamma+\gamma^s)$, as we have conjectured before.

C. Strong scattering

Let us now go beyond the weak scattering limit, so that we cannot any more use the expansion in $X\Lambda$ in Eq. (8). In accordance with the AG result, the critical temperature vanishes at some finite rate of intraband magnetic scattering, $\gamma^s_{\alpha\beta}\sim T_{c0}$. The situation is qualitatively different with respect to interband scattering. We will show that in the strongly anisotropic case of $\lambda_{11},\lambda_{22}\gg\lambda_{\alpha\neq\beta}$ the critical temperature does not vanish even in the regime of very strong interband scattering. Let us first consider the intermediate scattering regime $\pi T_c\ll\gamma_{\alpha\beta}\ll\omega_D$. In this case one still can use Eqs. (6)–(8). Using expansion

$$\chi(x\to\infty)=\ln(4\gamma^*x+\text{const}),$$

we obtain that

$$\Lambda_{\rm eff}=\Lambda-\Lambda\cdot\ln(\gamma^*\hat{g}^+/\pi T+\text{const})\cdot\Lambda,$$

which has a particularly simple form for the case we are interested in, $\gamma_{11}=\gamma_{22}=0$,

$$\hat{g}^{+}=\begin{pmatrix} \gamma_{12} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{21} \end{pmatrix}:$$

$$\Lambda_{\mathrm{eff}}=\Lambda-(\gamma_{12}+\gamma_{21})^{-1}\ln[4\gamma^{*}(\gamma_{12}+\gamma_{21})/\pi T_c]\Lambda\cdot\hat{g}^{+}\cdot\Lambda,$$

$$\frac{T_c}{T_{c0}}=\left[\frac{\pi T_c}{2\gamma^{*}(\gamma_{12}+\gamma_{21})}\right]^{\gamma_{12}/(\gamma_{12}+\gamma_{21})}, \qquad (18)$$

where we assumed, to be specific, that $\lambda_{11}\geqslant\lambda_{22}$. Solving for T_c, we get

$$\frac{T_c}{T_{c0}}=\left[\frac{\pi T_{c0}}{2\gamma^{*}(\gamma_{12}+\gamma_{21})}\right]^{\gamma_{12}/\gamma_{21}}=\left[\frac{\pi T_{c0}}{2\gamma^{*}(\gamma_{12}+\gamma_{21})}\right]^{N_2/N_1}, \qquad (19)$$

where $N_{1,2}$ are the densities of states (DOS) in the two bands (the last equality appears because $\gamma_{\alpha\beta}/\gamma_{\beta\alpha}=N_\beta/N_\alpha$).

Equation (19) gives the limit $T_c\to 0$ when $\gamma_{\alpha\beta}\to\infty$. However, Eq. (8) becomes invalid in this regime, namely, when the interband scattering rate $\gamma_{\alpha\beta}$ exceeds the characteristic electronic energy scale ω_D which is relevant for Cooper pairing. In this case, we have to go back to Eq. (1d). This equation can be solved analytically in an important regime of the isotropic superstrong interband scattering, $\gamma_{\alpha\beta}=\gamma(1-\delta_{\alpha\beta})N_\beta$. In this regime, $g^{+}_{\alpha\beta}=\gamma(\delta_{\alpha\beta}N-N_\beta)$, where $N=\Sigma_\alpha N_\alpha$. To handle Eq. (1d) we first need to transform the matrix

$$\begin{aligned}(2\omega_n\delta_{\alpha\beta}+g^{+}_{\alpha\beta})^{-1}&=[(2\omega_n+\gamma N)\delta_{\alpha\beta}-\gamma N_\beta]^{-1}\\ &=(2\omega_n+\gamma N)^{-1}[\delta_{\alpha\beta}-\gamma N_\beta/\\ &\quad\times(2\omega_n+\gamma N)]^{-1}\end{aligned}$$

to a more tractable form. Expanding the square brackets in series in $\gamma N_\beta/(2\omega_n+\gamma N)$ and collecting the appropriate terms, we observe that

$$(2\omega_n\delta_{\alpha\beta}+g^{+}_{\alpha\beta})^{-1}=(2\omega_n+\gamma N)^{-1}(\delta_{\alpha\beta}+\gamma N_\alpha/2\omega_n),$$

which in the sought limit $\gamma\to\infty$ is simply $N_\alpha/2N\omega_n$. Thus

$$\Delta_\alpha=\sum_\beta \Lambda_{\alpha\beta}\sum_k \frac{N_k\Delta_k}{N}\sum_n \frac{2\pi T}{\omega_n}, \qquad (20)$$

which has the solution

$$\Delta_\alpha=\lambda_\alpha\overline{\Delta}\sum_n \frac{2\pi T}{\omega_n}, \qquad (21)$$

where $\lambda_\alpha=\Sigma\Lambda_{\alpha\beta}$ is the mass renormalization parameter, and the average gap $\overline{\Delta}=\Sigma_\alpha N_\alpha\Delta_\alpha/N$ satisfies the regular BCS equation with isotropically averaged coupling:

$$\overline{\Delta}=\lambda\overline{\Delta}\sum_n \frac{2\pi T}{\omega_n}, \quad \lambda=\sum_\alpha N_\alpha\lambda_\alpha/N. \qquad (22)$$

So in the superstrong coupling regime T_c saturates at a limiting value, which is actually the critical temperature calculated in fully isotropic BCS theory. This regime corresponds to the so-called Cooper limit investigated previously for proximity-effect coupled systems.[15,24] Note that the order parameters in the individual bands are nevertheless different, specifically $\Delta_\alpha=\overline{\Delta}\lambda_\alpha/\lambda$. This does not mean that the observable zero-temperature gaps are going to be different. In fact, they are the same, as discussed in the next section and illustrated in Fig. 3.

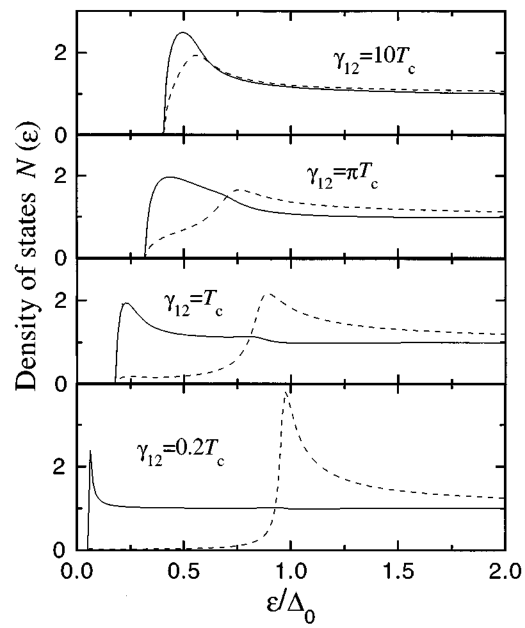

FIG. 3. Superconducting density of states in a two-band model with $\lambda_{12}=\lambda_{21}=\lambda_{22}=0$, $\lambda_{11}=0.5$. Only interband nonmagnetic scattering is included. The solid lines show the DOS in band 2 and the dashed lines in band 1. Note that both DOS coincide in the regime of strong scattering.

We illustrate the above discussion of the T_c suppression by a numerical solution of the Eq. (1d) for a two-band case with strong interband anisotropy and $\lambda_{12},\lambda_{21}\ll\lambda_{11},\lambda_{22}$, $\gamma^s_{21}=\gamma^s_{12}$, corresponding to equal densities of states in the two bands $N_1=N_2$. We have chosen $\lambda_1=0.2$, $\lambda_{12},\lambda_{21}=0$, and $\gamma^s_{21}=\gamma^s_{12}=0$. The results of calculations of T_c vs γ_{12} are shown in Fig. 1 for various values of λ_2. In accordance with the above analytical results, T_c first drops steeply as γ_{12} increases and then saturates at some finite value when $\gamma_{12}\sim\omega_D$. The saturation value depends on λ_2 in accordance with Eq. (22). The suppression of T_c remains the same when nonmagnetic scattering is zero, $\gamma_{12}=0$, but γ^s_{12} is finite, except that the order parameter $\Delta_{2,n=0}$ has the sign opposite to that of $\Delta_{1,n=0}$, as discussed above and illustrated in Fig. 2. Both kinds of impurities γ^s_{12} and γ_{12} suppress the critical temperature in this case according to Eqs. (14)–(22).

IV. DENSITY OF STATES AND SUPERCONDUCTING GAP

The discussion of the critical temperature suppression was based on the solutions of the linearized equations. To obtain the density of states, the nonlinear equations (1) should be solved. In the presence of impurities there is no distinct gap, in the sense that the minimal excitation energy does not coincide with the maximum in the density of states. The latter is defined in terms of $\Delta'_{n\alpha}$. Namely, the superconducting density of states $N(\omega)$ in units of the normal density of states at the Fermi level N_0 is

$$\frac{N(\omega)}{N_0} = \mathrm{Re}\frac{\omega}{\sqrt{\omega^2 - \Delta'^2(\omega)}},$$

where $\Delta'(\omega)$ is the analytical continuation of Δ'_n. An analytical solution for $\Delta'_{n\alpha}$ is not straightforward to obtain; however, some properties of the numerically obtained solutions for $\Delta'_{n\alpha}$ are already illustrated above in Fig. 2. Moreover, there are some rigorous statements that can be made about $\Delta'_{n\alpha}$.

Let us consider again the limit of superstrong isotropic nonmagnetic scattering. We have shown above that T_c in this case is reduced to T_c of the equivalent BCS system with the isotropic coupling constant. The same statement appears to be true for the gap in the excitation spectrum. Indeed, following AG, we can define $u_{\alpha n} = \omega_n / \Delta'_{in}$, and, after the usual transformation $\omega_n \to -i\omega$, $u_{\alpha n} \to -iu_\alpha$, $\Delta'_{\alpha n} \to \Delta'_\alpha(\omega)$, we can write down the multiband analog of Eq. (42′) of AG:

$$\frac{\omega}{\Delta_\alpha} = u_\alpha - \frac{1}{\Delta_\alpha}\sum_\beta \frac{u_\alpha \Gamma^+_{\alpha\beta} - \Gamma^-_{\alpha\beta} u_\beta}{\sqrt{1-u_\beta^2}}. \tag{23}$$

In the absence of magnetic impurities we let $\Gamma^+_{\alpha\beta} = \Gamma^-_{\alpha\beta} = \gamma$, and take the limit $\gamma \to \infty$. Evidently, a solution of Eq. (23) in this limit exists only if $u_\alpha = u_\beta$, and correspondingly Δ'_α does not depend on α. We conclude that in this limit the reduced density of states is the same in all bands, and in fact coincides with that of the isotropic BCS model with the gap determined from the nonlinear analog of Eq. (22): $\bar{\Delta} = \lambda \bar{\Delta} \Sigma_n 2\pi T / \sqrt{\omega_n^2 + \bar{\Delta}^2}$, with $\lambda = \Sigma_\alpha N_\alpha \lambda_\alpha / N$.

The evolution of the densities of states in the multiband nonmagnetic scattering case is shown in Fig. 3. Here, we show the results of numerical solution of Eqs. (1) in the weak coupling regime with $\lambda_1 = 0.5$, $\lambda_{12}, \lambda_{21}, \lambda_{22} = 0$, and $\gamma^s_{\alpha\beta} = 0$. Only nonmagnetic interband scattering $\gamma_{12} = \gamma_{21}$ is included. In the clean limit, the two bands show two different excitation gaps. In accordance with earlier calculations,[24,25] any weak, but finite impurity scattering mixes the pairs in the two bands, so that the first band (with the larger gap, i.e., more superconducting) develops a tail in the density of states which extends all the way down to the second-band gap. Except for this tail, which consists of the normal excitations of the second band, scattered into the first band by impurities, the density of states still looks similar to the clean-limit one. Upon the increase of the scattering rate, the low-energy tail in the first-band density of states grows, and the minimal gap, the gap in the second band, grows as well. This reflects the fact that a larger number of pairs is scattered into the second band and induced the interband-proximity-effect superconductivity there. Thus the *decrease* in the critical temperature of the system is accompanied by an *increase* of the minimal gap in the excitation spectrum.

Next, let us include interband magnetic scattering into Eq. (23). Then in the considered regime $\gamma \to \infty$ we have $\Gamma^+_{\alpha\beta} = \gamma + \gamma^s$, $\Gamma^-_{\alpha\beta} = \gamma - \gamma^s$, and $u_\alpha = u_\beta \equiv u$ (we assumed $\gamma^s_{\alpha\neq\beta} \equiv \gamma^s$). As a result, the densities of states in each band are given by $N(\omega) = \mathrm{Re}\, u / \sqrt{u^2 - 1}$, where u is a solution of the equation

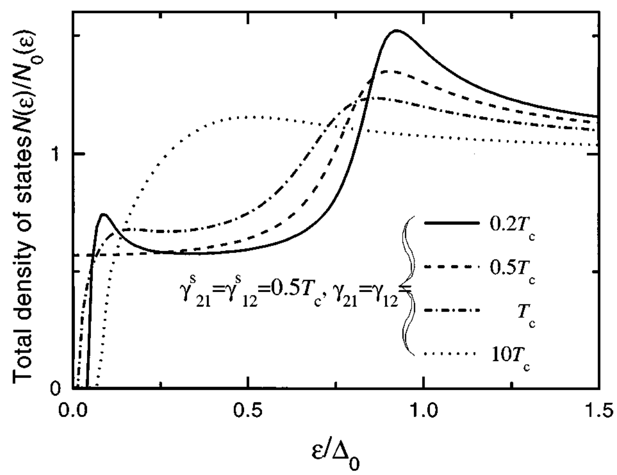

FIG. 4. Total superconducting density of states in a two-band model with $\lambda_{12} = \lambda_{21} = \lambda_{22} = 0$, $\lambda_{11} = 0.5$.

$$\frac{\omega}{\Delta} = u\left(1 - \frac{2\gamma^s}{\Delta}\frac{1}{\sqrt{1-u^2}}\right). \tag{24}$$

An energy gap corresponds to a maximum real solution for u in the interval $u < 1$ and the pair-breaking rate is given by $2\gamma^s$. Thus, with an increase of nonmagnetic scattering we have a crossover from the state with different signs of order parameters in different bands (for zero γ) to the isotropic state (for $\gamma \to \infty$). This isotropic state may be normal, gapless, or gapped, depending on the value of $2\gamma^s$. Following the AG analysis, we obtain that an energy gap at $\gamma \to \infty$ will exist if $\gamma^s < \exp(-\pi/4)\Delta_0/2$, where Δ_0 is the BCS gap at $T = 0$. This case is particularly interesting: Since in the isotropic ($\gamma \to \infty$) limit there is a finite gap and a finite positive order parameter in both bands, and in the opposite limit of small γ the order parameter in one band is negative, it is clear that at intermediate values of interband scattering γ a gapless state should be crossed over. The last statement is in agreement with the result of Ref. 1 that for an order parameter with a sign change and a nonzero Fermi-surface average, a gapless state develops with an increase of impurity concentration, but the gap is restored at a large concentration of impurities. To illustrate such a crossover, induced by magnetic scattering, we show in Fig. 4 the results of our numerical calculations for a weak coupling two-band model ($\lambda_1 = 0.5, \lambda_{12}, \lambda_{21}, \lambda_{22} = 0$, and $\gamma^s_{12} = \gamma^s_{21} = T_c/2$). The total density of states is shown at different values of the interband nonmagnetic scattering rate γ. According to the discussion in Sec. ISR, for a small γ both order parameters have different signs. In this case γ^s has no pair breaking effect, and small gap (negative order parameter) is induced in the second band by the magnetic scattering. On the contrary, the interband γ is in this situation pair breaking. The shape of the density of states shows two characteristic peaks, one, at about $0.1\ \Delta_0$, due to this induced energy gap in the second band and another just below Δ_0 from the gap in the first band.

With the increase of the nonmagnetic scattering rate the order parameter in the second band becomes smaller in absolute value, still remaining negative. The lower peak in the density of states gets washed out and the minimal energy gap becomes smaller. When γ approaches γ^s this small gap vanishes, although there is still a distinguishable peak in the density of states coming from the gap in the first band. At larger $\gamma \gg \gamma^s$ both gaps have again the same sign, and now it is γ^s, which is pair breaking. As one again can see from Fig. 4, a small gap is restored for the last two curves, corresponding to $\gamma = 2\gamma^s = T_c$ and $\gamma = 20\gamma^s = 10T_c$. Note that at $\gamma \gg T_c$, the gap cannot any more be ascribable to any of the two bands, but corresponds to a fully isotropic superconductivity, as described by Eqs. (14)–(22).

V. CONCLUSIONS

In conclusion, we generalized the Abrikosov-Gor'kov solution to the case of arbitrary interband anisotropy of the pairing interaction, and arbitrary strength and anisotropy of magnetic and/or nonmagnetic impurity scattering. The results are illustrated on model two-band systems with interband anisotropy and various kinds of impurity scattering. In the case of weak scattering, we found an analytic solution, analogous to the isotropic solution of Abrikosov and Gor'kov. For weak anisotropy, this solution yields the critical temperature suppression proportional to the mean square variation of the order parameter, a fact earlier pointed out by several authors in various special cases. We also proved analytically that the superconductivity suppression by isotropic magnetic and isotropic nonmagnetic impurities is exactly the same when the average order parameter is zero (e.g., in case of *d* pairing). We also give an analytical solution for T_c in the two-band model in the case of intermediate strength scattering. In the case of superstrong scattering we find a solution for T_c for arbitrary anisotropy. We also discussed the evolution of the density of states with the increase of the impurities concentration (or scattering strength).

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