Lecture 12

Casella and Berger

Section 10.3

Asymptotics in Hypothesis Testing

Theorem 10.3.1 (Asymptotic distribution of the LRT) For testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, suppose $X_1, ..., X_n$ are i.i.d. $f(x|\theta)$, $\hat{\theta}$ is the MLE of θ, and $f(x|\theta)$ satisfies the regularity conditions in Miscellanea 10.6.2. Then under H_0, as $n \to \infty$,

$$-2 \log \lambda(\mathbf{X}) \to \chi_1^2 \text{ in distribution.}$$

Example 10.3.2 (Poisson LRT) Suppose $X_1, ..., X_n$ i.i.d. Poisson(λ). Consider testing $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$.

Theorem 10.3.3 Let $X_1, ..., X_n$ be a random sample from a pdf or pmf $f(x|\theta)$. Under the regularity conditions in Miscellanea 10.6.2, if $\theta \in \Theta_0$, then the distribution of the statistic $-2\log\lambda(\mathbf{X})$ converges to a chi square distribution as the sample size $n \to \infty$. The degrees of freedom of the limiting distribution is the difference between the number of free parameters specified by $\theta \in \Theta_0$ and the number of free parameters specified by $\theta \in \Theta$.

Rejection of $H_0 : \theta \in \Theta_0$ for small values of $\lambda(\mathbf{X})$ is equivalent to rejection for large values of $-2\log\lambda(\mathbf{X})$. Thus,

$$H_0 \text{ is rejected if and only if } -2\log\lambda(\mathbf{X}) \geq \chi^2_{\nu,\alpha},$$

where ν is the degrees of freedom specified in Theorem 10.3.3.

Example 10.3.4 (Multinomial LRT) Let $\theta = (p_1, ..., p_5)$, where the p_js are nonnegtive and sum to 1. Suppose $X_1, ..., X_n$ are i.i.d. discrete random variables and $P_\theta(X_i = j) = p_j, j = 1, ..., 5$. Consider testing

$$H_0 : p_1 = p_2 = p_3 \text{ and } p_4 = p_5 \text{ versus } H_1 : H_0 \text{ is not true.}$$

Wald Test

In general, a *Wald test* is a test based on a statistic of the form

$$Z_n = \frac{W_n - \theta_0}{S_n},$$

where θ_0 is a hypothesized value of the parameter θ, W_n is an estimator of θ, and S_n is a standard error for W_n, an estimator of the standard deviation of W_n. For example, if W_n is the MLE of θ, then $S_n = 1/\sqrt{\hat{I}_n(W_n)}$, where

$$\hat{I}_n(W_n) = -\frac{\partial^2}{\partial\theta^2} \log L(\theta|\mathbf{X})|_{\theta=W_n}$$

is the observed information number.

Under H_0, $Z_n \to N(0,1)$ in distribution.

Example 10.3.5 (Large-sample binomial tests) Let $X_1, ..., X_n$ be a random sample from a Bernoulli(p) population. Consider testing $H_0 : p \leq p_0$ versus $H_1 : p > p_0$, where $0 < p_0 < 1$ is a specified value.

Score Test

Score statistic $S(\theta)$

$$S(\theta) = \frac{\partial}{\partial\theta} \log L(\theta|\mathbf{X}).$$

It can be shown that

$$E_\theta S(\theta) = 0$$

and

$$\mathrm{Var}_\theta S(\theta) = E_\theta\left(\left(\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{X})\right)^2\right) = -E_\theta\left(\frac{\partial^2}{\partial\theta^2}\log L(\theta|\mathbf{X})\right) = I_n(\theta).$$

The test statistic for the score test is

$$Z_S = S(\theta_0)/\sqrt{I_n(\theta_0)}.$$

Under H_0, $Z_S \to N(0,1)$ in distribution.

Example 10.3.6 (Binomial score test)