Lecture 11

Casella and Berger

Sections 8.3.2, 8.3.4

Unbiased Test A test with power function $\beta(\theta)$ is *unbiased* if $\beta(\theta') \geq \beta(\theta'')$ for every $\theta' \in \Theta_0^c$ and $\theta'' \in \Theta_0$.

In most problems there are many unbiased tests. How to determine which one is better?

Most Powerful Tests

Definition 8.3.11 Let $\mathcal{C}$ be a class of tests for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_0^c$. A test in class $\mathcal{C}$, with power function $\beta(\theta)$, is a *uniformly most powerful* (UMP) class $\mathcal{C}$ test if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_0^c$ and every $\beta'(\theta)$ that is a power function of a test in class $\mathcal{C}$.

If the class $\mathcal{C}$ is the class of all level α tests, then the test described above is called a UMP level α test.

Three types of hypotheses
Simple hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$
One-sided hypotheses $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$
Two-sided hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$

Theorem 8.3.12 (Neyman-Pearson Lemma) Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where the pdf or pmf corresponding to θ_i is $f(\mathbf{x}|\theta_i), i = 0, 1$, using a test with rejection region R that satisfies

$$\mathbf{x} \in R \text{ if } f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0)$$

and

$$\mathbf{x} \in R^c \text{ if } f(\mathbf{x}|\theta_1) < kf(\mathbf{x}|\theta_0), \tag{8.3.1}$$

for some $k \geq 0$, and

$$\alpha = P_{\theta_0}(\mathbf{X} \in R). \tag{8.3.2}$$

Then

a. (Sufficiency) Any test that satisfies (8.3.1) and (8.3.2) is a UMP level α test.

b. (Necessity) If there exists a test satisfying (8.3.1) and (8.3.2) with $k > 0$, then every UMP level α test is a size α test (satisfies (8.3.2)) and every UMP level α test satisfies (8.3.1) except perhaps on a set A satisfying $P_{\theta_0}(\mathbf{X} \in A) = P_{\theta_1}(\mathbf{X} \in A) = 0$.

Corollary 8.1.3 Extension of Neyman-Pearson Lemma to tests based on sufficient statistics.

Example 8.3.15 (UMP normal test) Let $X_1, ..., X_n$ be a random sample from a $N(\theta, \sigma^2)$ population, σ^2 known. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_0 > \theta_1$. Find the UMP level α test.

UMP level α test for one-sided hypotheses

Definition 8.3.16 A family of pdfs or pmfs $\{g(t|\theta) : \theta \in \Theta\}$ for a univariate random variable T with real-valued parameter θ has a *monotone likelihood ratio* (MLR) if, for every $\theta_2 > \theta_1, g(t|\theta_2)/g(t|\theta_1)$ is a monotone function of t on $\{t : g(t|\theta_1) > 0$ or $g(t|\theta_2) > 0\}$. Note that $c/0$ is defined as ∞ if $0 < c$.

Any regular exponential family with $g(t|\theta) = h(t)c(\theta)e^{w(\theta)t}$ has an MLR if $w(\theta)$ is a nondecreasing function.

Theorem 8.3.17 (Karlin-Rubin) Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Suppose T is a sufficient statistic for θ and the family of pdfs or pmfs $\{g(t|\theta) : \theta \in \Theta\}$ of T has an MLR. Then for any t_0, the test rejects H_0 if and only if $T > t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$. Similarly, the test that rejects $H_0 : \theta \geq \theta_0$ in favor of $H_1 : \theta < \theta_0$ if and only if $T < t_0$ is a UMP level $\alpha = P_{\theta_0}(T < t_0)$ test.

Example 8.3.18 (Continuation of Example 8.3.15) Consider testing $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$. Find the UMP level α test.

Example 8.3.19 (Nonexistence of UMP test, continuation of Example 8.3.15) Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Then the UMP level α test does not exist.

Example 8.3.20 (Unbiased test) When no UMP level α test exists within the class of all tests, we might try to find a UMP level α within the class of unbiased tests.

p-Values

Definition 8.3.26 A *p-value* $p(\mathbf{X})$ is a test statistic satisfying $0 \leq p(\mathbf{x}) \leq 1$ for every sample point $\mathbf{x}$. Small values of $p(\mathbf{X})$ give evidence that H_1 is true. A p-value is *valid* if, for every $\theta \in \Theta_0$ and every $0 \leq \alpha \leq 1$,

$$P_\theta(p(\mathbf{X}) \leq \alpha) \leq \alpha.$$

Theorem 8.3.27 Let $W(\mathbf{X})$ be a test statistic such that large values of W give evidence that H_1 is true. For each sample point $\mathbf{x}$, define

$$p(\mathbf{x}) = \sup_{\theta \in \Theta_0} P_\theta(W(\mathbf{X}) \geq W(\mathbf{x})).$$

Then, $p(\mathbf{X})$ is a valid p-value.

Example 8.3.28 (Two-sided normal p-value) Let $X_1, ..., X_n$ be a random sample from a $N(\mu, \sigma^2)$ population, σ^2 unknown. Consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. The LRT rejects H_0 for large values of $W(\mathbf{X}) = |\bar{X} - \mu_0|/(S/\sqrt{n})$. For a realization of the random sample $\mathbf{x}$, find the p-value.

Example 8.3.29 (One-sided normal p-value) Consider testing $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$.

Example 8.3.30 (Fisher's Exact Test) Let S_1 and S_2 be independent observations with $S_1 \sim binomial(n_1, p_1)$ and $S_2 \sim binomial(n_2, p_2)$. Consider testing $H_0 : p_1 = p_2$ versus $H_1 : p_1 > p_2$.