Lecture 8

Casella and Berger

Sections 7.3.3, 10.1.1, 10.1.2, 10.1.3, 10.2.1

Sufficiency and Unbiasedness

Theorem 8.1 (Rao-Blackwell) Let W be any unbiased estimator of $\tau(\theta)$, and let T be a sufficient statistic for θ. Define $\phi(T) = E(W|T)$. Then $E_\theta \phi(T) = \tau(\theta)$ and $\mathrm{Var}_\theta \phi(\mathrm{T}) \leq \mathrm{Var}_\theta \mathrm{W}$ for all θ; that is, $\phi(T)$ is a uniformly better unbiased estimator of $\tau(\theta)$.

Theorem 8.2 If W is a best unbiased estimator of $\tau(\theta)$, then W is unique.

Theorem 8.3 If $E_\theta W = \tau(\theta)$, W is the best unbiased estimator of $\tau(\theta)$ if and only if W is uncorrelated with all unbiased estimators of 0.

Theorem 8.4 Let T be a complete sufficient statistic for a parameter θ, and let $\phi(T)$ be any estimator based only on T. Then $\phi(T)$ is the unique best unbiased estimator of its expected value.

Asymptotic Properties of Point Estimators

- Consistency
- Asymptotic Efficiency
- Asymptotic Normality

Definition 8.1 (Consistency) A sequence of estimators $W_n = W_n(X_1, ..., X_n)$ is a consistent sequence of estimators of the parameter θ if, for every $\epsilon > 0$ and every $\theta \in \Theta$,

$$\lim_{n\to\infty} P_\theta(|W_n - \theta| < \epsilon) = 1.$$

Example 8.1 (Consistency of $\bar{X}_n$) Let $X_1, X_2, ...$ be i.i.d. $\sim N(\theta, 1)$, then $\bar{X}_n$ is a consistent sequence of estimators of θ.

A sufficient condition

$$\lim_{n\to\infty} E_\theta[(W_n - \theta)^2] = 0.$$

Theorem 8.5 If W_n is a sequence of estimators of a parameter θ satisfying

i. $\lim_{n\to\infty} Var_\theta W_n = 0$,

ii. $\lim_{n\to\infty} Bias_\theta W_n = 0$,

for every $\theta \in \Theta$, then W_n is a consistent sequence of estimators of θ.

Theorem 8.6 Let W_n be a consistent sequence of estimators of a parameter θ. Let $a_1, a_2, ...$ and $b_1, b_2, ...$ be sequences of constants satisfying

i. $\lim_{n \to \infty} a_n = 1$,

ii. $\lim_{n \to \infty} b_n = 0$.

Then the sequence $U_n = a_n W_n + b_n$ is a consistent sequence of estimators of θ.

Theorem 8.7 (Consistency of MLEs) Let $X_1, X_2, ...$ be i.i.d. $\sim f(x|\theta)$, and let $L(\theta|\mathbf{x})$ be the likelihood function. Let $\hat{\theta}_n$ denote the MLE of θ. Let $\tau(\theta)$ be a continuous function of θ. Under certain regularity conditions (see 10.6.2) on $f(x|\theta)$, $\tau(\hat{\theta}_n)$ is a consistent estimator of$\tau(\theta)$.

Consistency: concerned with the asymptotic accuracy of an estimator

Efficiency: concerned with the asymptotic variance of an estimator

In many cases, for an estimator T_n, $VarT_n \to 0$.

Definition 8.2 (Limiting variance) For an estimator T_n, if $\lim_{n\to\infty} k_n VarT_n = \tau^2 < \infty$, where $\{k_n\}$ is a sequence of constants, then τ^2 is called the *limiting variance* or *limit of the variances.*

Example 8.2 (Limiting variances) Let $X_1, X_2, ...$ i.i.d. $\sim N(\mu, \sigma^2)$. Find the limiting variances of $\bar{X}_n$ and $1/\bar{X}_n$.

Definition 8.3 (Asymptotic variance) For an estimator T_n, suppose that $k_n(T_n - \tau(\theta)) \to N(0, \sigma^2)$ in distribution. The parameter σ^2 is called the *asymptotic variance* or *variance of the limit distribution* of T_n.

Definition 8.4 (Asymptotic efficiency) A sequence of estimators W_n is *asymptotically efficient* for a parameter $\tau(\theta)$ if $\sqrt{n}[W_n - \tau(\theta)] \to N(0, v(\theta))$ in distribution and

$$v(\theta) = \frac{[\tau'(\theta)]^2}{E_\theta[(\frac{\partial}{\partial\theta} \log f(X|\theta))^2]};$$

that is, the asymptotic variance of W_n achieves the Cramer-Rao Lower Bound.

Theorem 8.8 (Asymptotic efficiency of MLEs) Under certain regularity conditions (see 10.6.2),

$$\sqrt{n}[\tau(\hat{\theta}_n) - \tau(\theta)] \to N(0, v(\theta)),$$

where $v(\theta)$ is the Cramer-Rao Lower Bound. That is, $\tau(\hat{\theta}_n)$ is a consistent and asymptotic efficient estimator of $\tau(\theta)$.

Example 8.3 (Asymptotic normality and consistency)
Asymptotic normality implies consistency.

How to estimate asymptotic variance $Var_\theta h(\hat{\theta}_n)$?
1. approximate $Var_\theta h(\hat{\theta}_n)$ by $[h'(\theta)]^2 Var_\theta \hat{\theta}_n$.
2. estimate the resulting approximation, usually by substituting $\hat{\theta}_n$ for θ.

Example 8.4 (Approximate binomial variance) Let $X_1, ..., X_n$ from a Bernoulli(p) population. Estimate the asymptotic variance of $\hat{p}_n = \bar{X}_n$ and $\frac{\hat{p}_n}{1-\hat{p}_n}$.

Definition 8.5 (Asymptotic Relative Efficiency) If two estimators W_n and V_n satisfy

$$\sqrt{n}[W_n - \tau(\theta)] \to N(0, \sigma_W^2)$$

and

$$\sqrt{n}[V_n - \tau(\theta)] \to N(0, \sigma_V^2)$$

in distribution, the *asymptotic relative efficiency* (ARE) of V_n w.r.t. W_n is

$$\mathrm{ART}(V_n, W_n) = \frac{\sigma_W^2}{\sigma_V^2}.$$

Example 8.5 (AREs of the median to the mean)