Solutions for
Midterm Exam 1
STAT 544, Spring 2018

Instructions: You cannot use any books or notes. You can use a calculator, but not a computer. You are not allowed to communicate with anyone (verbally, in writing, or electronically), except for me, during the exam period (and at most I will clarify a problem — I won't give you any help with solving the problems). Phones should be kept out of your hands while you're taking the exam.

There are 8 problems having a total of 11 parts. Each part is worth 10 points, and your best 10 of 11 scores will be summed to obtain your score for this exam. To receive full credit, you need to **justify your answers**, using notation and terminology properly, and clearly defining any events that you use (that aren't defined in the statement of the problem).

Express probabilities as exact values (as fractions, or in decimal form), or else round them to three significant digits. (Note: 0.00402 has three significant digits, and 0.004 only has one significant digit.) Do not express final answers as expressions that need to be evaluated. (Your final answers should not include any letters, binomial coefficients, or even factorials.)

Put all of your work on these sheets. If you need more room, direct me to look for additional work on the back of a page. **Draw boxes around your final answers!**

1) Consider a set of nine balls having the numbers 1 through 9 on them (one number per ball). If a subset of three of the nine balls is randomly selected, what is the probability that at least one of the three numbers selected is an odd integer? (That is, what is the probability that at least one of the three numbers selected belong to the set $\{1, 3, 5, 7, 9\}$?)

$$P(\text{at least one odd}) = 1 - P(\text{no odd})$$
$$= 1 - \frac{\binom{5}{0}\binom{4}{3}}{\binom{9}{3}}$$
$$= 1 - \frac{1 \cdot 4}{84}$$
$$= \boxed{\frac{20}{21} \; (\doteq 0.952)}.$$

2) A, B, and C are events for which $P(A) = 0.3$, $P(B) = 0.4$, and $P(C) = 0.5$. If A and B are mutually exclusive, A and C are independent, and $P(B|C) = 0.1$, what is the value of $P(A \cup B \cup C)$?

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
$$\overset{ind.}{=} P(A) + P(B) + P(C) - P(\emptyset) - P(A)P(C) - P(B|C)P(C) + P(\emptyset)$$
$$= 0.3 + 0.4 + 0.5 - 0 - (0.3)(0.5) - (0.1)(0.5) + 0$$
$$= \boxed{1}.$$

4 scores in [93.6, 99.7]
4 scores in [72.3, 79.7]
1 score below 40

3) Suppose that a ball will be randomly drawn from an urn initially containing 5 amber, 5 blue, and 5 green balls. That ball will not be replaced in the urn, and furthermore two more balls of the same color will also be removed from the urn. Then another ball will be randomly selected from the 12 balls which remain in the urn. (So, for example, if a green ball is initially chosen, two additional green balls will also be removed from the urn, so that at the time a second ball is randomly drawn, the urn will contain 5 amber, 5 blue, and only 2 green balls.)

(a) What is the probability that the second ball selected will be blue?

Letting B_i be the event that a blue ball is chosen on draw i, the desired probability is

$$P(B_2) = P(B_2 \mid B_1)P(B_1) + P(B_2 \mid B_1^c)P(B_1^c)$$
$$= \left(\tfrac{2}{12}\right)\left(\tfrac{5}{15}\right) + \left(\tfrac{5}{12}\right)\left(\tfrac{10}{15}\right)$$
$$= \boxed{\tfrac{1}{3}}.$$

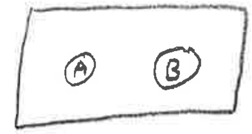
(This value also follows from symmetry since each of the 15 balls is equally likely to be the 2nd ball drawn.)

(b) Given that the second ball selected is blue, what is the probability that the first ball selected is blue?

Using the events from part (a), the desired conditional prob. is

Bayes
$$P(B_1 \mid B_2) = \frac{P(B_1 \cap B_2)}{P(B_2)}$$
$$= \frac{P(B_2 \mid B_1)\,P(B_1)}{P(B_2)}$$

using answer from part (a)
$$= \frac{\left(\tfrac{2}{12}\right)\left(\tfrac{5}{15}\right)}{\left(\tfrac{1}{3}\right)}$$
$$= \boxed{\tfrac{1}{6}}.$$



4) A and B are mutually-exclusive events with $P(A) = 0.1$ and $P(B) = 0.2$.
(a) What is the value of $P(A^C \cup B^C)$?

$$P(A^c \cup B^c) \overset{\text{DeMorgan}}{=} P((A \cap B)^c)$$
$$= 1 - P(A \cap B)$$
$$= 1 - P(\emptyset)$$
$$= 1 - 0$$
$$= \boxed{1}.$$

(b) What is the value of $P(A^C \cap B^C)$?

$$P(A^c \cap B^c) \overset{\text{DeMorgan}}{=} P((A \cup B)^c)$$
$$= 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B)]$$
$$= 1 - [0.1 + 0.2]$$
$$= \boxed{0.7}.$$

5) A and B are independent events with $P(A) = 0.1$ and $P(B) = 0.2$.
(a) What is the value of $P(A^C \cup B^C)$?

$$P(A^c \cup B^c) \overset{\text{DeMorgan}}{=} P((A \cap B)^c)$$
$$= 1 - P(A \cap B)$$
$$\overset{\text{ind.}}{=} 1 - P(A)\,P(B)$$
$$= 1 - (0.1)(0.2)$$
$$= \boxed{0.98}.$$

(Alternatively, this equals

$$P(A^c) + P(B^c) - P(A^c \cap B^c)$$
$$= [1 - P(A)] + [1 - P(B)] - [1 - P(A)][1 - P(B)]$$
$$= 0.9 + 0.8 - (0.9)(0.8)$$
$$= 0.98.)$$

(b) What is the value of $P(A^C \cap B^C)$?

Since

A & B are ind. $\Rightarrow$ A^c & B^c ind.,

we have

$$P(A^c \cap B^c) \overset{\text{ind}}{=} P(A^c)\,P(B^c)$$
$$= [1 - P(A)][1 - P(B)]$$
$$= [1 - 0.1][1 - 0.2]$$
$$= \boxed{0.72}.$$

6) **A fair die will be rolled three times. Making the usual assumptions of independence and equally-likely outcomes for each roll, what is the probability that three different faces will land upwards on the three rolls?**

Letting D_i be the event that roll i results in a different face than all previous rolls, the desired probability is

$$P(D_2 \cap D_3) = P(D_3 \mid D_2)\, P(D_2)$$
$$= \left(\tfrac{4}{6}\right)\left(\tfrac{5}{6}\right)$$
$$= \boxed{\tfrac{5}{9}}.$$

Alternatively, one can consider a sample space having $6^3 = 216$ equally-likely outcomes. Since there are $6 \cdot 5 \cdot 4$ ways to have all three rolls result in a different face upwards, we have that the desired probability is

$$\frac{6 \cdot 5 \cdot 4}{6^3} = \frac{20}{36} = \frac{5}{9}.$$

7) **Consider an urn that initially contains 100 balls.**
60 of them are colored the same shade of light gray.
30 of them are colored the same shade of medium gray.
10 of them are colored the same shade of dark gray.

A ball will be randomly drawn from this urn, and then a second ball will be randomly drawn from the 99 balls that remain after the first ball is drawn. What is the probability that the second ball will be darker than the first ball, given that the first ball is medium gray?

If a medium gray ball is drawn first, at the time of the 2nd draw, 10 of the remaining 99 balls will be darker. So a simple reduced sample space point of view gives us that the desired probability is $\boxed{\tfrac{10}{99}}$.

8) **Suppose that Angie, Brittany, Chloe, Denise, and Eve will be randomly ordered from left to right. What is the probability that all five women will be put in alphabetical order from left to right given that Brittany will be put somewhere to the left of Denise?**

Letting A be the event that the women are put in alphabetical order, we have

$$P(A) = \frac{1}{5!} = \frac{1}{120},$$

since only 1 of $5!$ possible orderings corresponds to alphabetical order (and all possible orderings are equally likely)

Letting B be the event that Brittany is put to the left of Denise, we have

$$P(B) = \frac{1}{2}$$

by symmetry (since each of these two are equally likely to be the leftmost one in a random ordering),

For the desired conditional probability, we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \overset{\text{since } A \subset B}{=} \frac{P(A)}{P(B)} = \frac{\frac{1}{120}}{\frac{1}{2}} = \boxed{\frac{1}{60}}.$$