Solutions for
Midterm Exam 2
STAT 544, Spring 2020

Instructions: This is an open book and open notes exam; you can use any written or printed materials that you have brought with you, but you are not allowed to share any materials during the exam period. You are not allowed to communicate with anyone (verbally, in writing, or electronically), except for me, during the exam period (and at most I will clarify the statement of a problem; I won't give you any hints about how to solve a problem). You can use a calculator and/or a computer if you wish to. Cell phones should be kept out of your hands during the exam period.

The 11 parts of the 5 problems are weighted equally, and I'll count your best 10 (of 11) scores from these parts.

Unless stated otherwise, for all parts of the exam you are to justify your answers, using notation and terminology properly, and clearly defining any events and random variables that you use (that aren't defined in the statement of the problem).

Express probabilities, expected values, variances, etc. as exact values (as fractions, integers, or in decimal form), or else round them to the nearest thousandth. Express them numerically — do not express final answers as expressions that need to be evaluated. (E.g., the standard normal cdf or the gamma function should not appear in any of your final answers.)

Put all of your work on these sheets. If you need more room, direct me to look for additional work on the back of a page. **Draw boxes around your final answers!**

1) X is a random variable having pmf

$$p_X(x) = \frac{2^x}{7} I_{\{0,1,2\}}(x).$$

(a) Letting F_X denote the cdf of X, give the value of $F_X(1.5)$.

$F_X(1.5) = P(X \le 1.5) = p_X(0) + p_X(1) = \frac{1}{7} + \frac{2}{7} = \boxed{\frac{3}{7}}$.

(b) Give the value of $E(X)$.

$E(X) = \sum_x x\, p_X(x) = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) = 0(\frac{1}{7}) + 1(\frac{2}{7}) + 2(\frac{4}{7}) = \boxed{\frac{10}{7}}$.

2) If X is a random variables for which $E(X) = 7$ and $Var(3X - 3) = 45$, what is the value of $E(X^2)$?

$45 = Var(3X-3) = 3^2 Var(X) \Rightarrow Var(X) = 5.$

$E(X^2) = Var(X) + [E(X)]^2 = 5 + 7^2 = \boxed{54}$.

3) Suppose that a continuously operating machine malfunctions randomly, according to a Poisson process having a rate of 5 malfunctions per week (7 days).

(a) What is the probability that exactly 1 of the next 7 days has no malfunctions?

The number of malfunctions in a day is a Poisson r.v. having mean

$$\left(\frac{5}{\text{week}}\right)\left(\frac{1}{7}\text{ week}\right) = \frac{5}{7}.$$

Letting p denote the prob. of no malfunctions in a day, we have

$$p = \frac{\left(\frac{5}{7}\right)^0 e^{-5/7}}{0!} = e^{-5/7}.$$

Letting Y be the number of the next 7 days w/ no malfunctions,

$$Y \sim \text{binomial}\,(7, p),$$

and the desired prob. is

$$P(Y=1) = \binom{7}{1} p(1-p)^6 = 7e^{-5/7}\left(1-e^{-5/7}\right)^6 \doteq \boxed{0.0606}.$$

(b) What is the probability that the 1st malfunction of a week occurs sometime during the 2nd day of the week?

Letting X be the time, in weeks, until the 1st malfunction, X is an exponential r.v. having mean $1/5$, and the desired prob. is

$$P\left(\tfrac{1}{7} < X < \tfrac{2}{7}\right) = \int_{1/7}^{2/7} 5e^{-5x}\,dx = -e^{-5x}\Big|_{1/7}^{2/7} = e^{-5/7} - e^{-10/7} \doteq \boxed{0.250}.$$

(Alternatively, letting V be the day number of the 1st malfunction, V is a geometric r.v., and the desired prob. is

$$P(V=2) = p(1-p) = e^{-5/7}\left(1-e^{-5/7}\right) \doteq 0.250$$

(where the value of p is obtained in part (a)).)

4) Consider a random variable X having cdf

$$F_X(x) = \begin{cases} 1, & x \geq 64, \\ \frac{x^{1/3}}{4}, & 0 < x < 64, \\ 0, & x \leq 0. \end{cases}$$

(a) Give the pdf of X.

Differentiating the cdf, one obtains

$$f_X(x) = \boxed{\frac{1}{12\, x^{2/3}} I_{(0,64)}(x)}$$

(b) Give the value of $P(X \geq 27)$.

$$P(X \geq 27) = 1 - P(X < 27) = 1 - F_X(27) = 1 - \frac{\sqrt[3]{27}}{4} = 1 - \frac{3}{4} = \boxed{\frac{1}{4}}.$$

(c) Letting U be a uniform (0, 1) random variable, give a function of U which has the same distribution as X.

For $x \in (0,1)$,

$$x = F_X(F_X^{-1}(x)) = \frac{\sqrt[3]{F_X^{-1}(x)}}{4}$$

$$\Rightarrow F_X^{-1}(x) = (4x)^3 = 64x^3.$$

So the desired function is $\boxed{64U^3}$.

(d) Give the cdf of $Y = X^{2/3}$.

The support of Y is $(0, 16)$. So $F_Y(y) = 0$ for $y \le 0$, and $F_Y(y) = 1$ for $y \ge 16$.

For $y \in (0, 16)$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X^{2/3} \le y) \\ &= P(X \le y^{3/2}) \\ &= F_X(y^{3/2}) \\ &= \frac{\sqrt{y}}{4}. \end{aligned}$$

Altogether, we have

$$\boxed{F_Y(y) = \begin{cases} 1, & y \ge 16, \\ \frac{\sqrt{y}}{4}, & 0 < y < 16, \\ 0, & y \le 0. \end{cases}}$$

(e) Give the value of $E(X^{2/3})$.

$$\begin{aligned} E(X^{2/3}) &= \int_{-\infty}^{\infty} x^{2/3} f_X(x)\,dx \\ &= \int_0^{64} x^{2/3} \frac{1}{12x^{2/3}}\,dx \\ &= \int_0^{64} \frac{1}{12}\,dx \\ &= \frac{x}{12}\Big|_0^{64} \\ &= \boxed{\frac{16}{3}}. \end{aligned}$$

As a check:

$$\begin{aligned} E(X^{2/3}) &= E(Y) \\ &= \int_{-\infty}^{\infty} y\, f_Y(y)\,dy \\ &= \int_0^{16} y \frac{1}{8\sqrt{y}}\,dy \\ &= \int_0^{16} \frac{1}{8} y^{1/2}\,dy \\ &= \frac{1}{12} y^{3/2}\Big|_0^{16} \\ &= \frac{16}{3}. \end{aligned}$$

5) Letting X be a normal random variable having mean μ and variance σ^2, express the 49th percentile of X's distribution, $\xi_{0.49}$, as a linear function of μ and σ, rounding the coefficient of σ to the nearest thousandth. (*Note*: $\xi_{0.49}$ is a value that satisfies $P(X < \xi_{0.49}) = 0.49$ and $P(X > \xi_{0.49}) = 0.51$.)

We have

$$0.49 = P(X \le \xi_{0.49})$$
$$= \Phi\left(\frac{\xi_{0.49} - \mu}{\sigma}\right)$$
$$\Rightarrow \frac{\xi_{0.49} - \mu}{\sigma} = \Phi^{-1}(0.49).$$

Since $\Phi(-0.02) \doteq 0.4920$ and $\Phi(-0.03) = 0.4880$), linear interpolation suggests that $\Phi(-0.025) \doteq 0.49$, which implies that $\Phi^{-1}(0.49) = -0.025$.

It then follows from above that

$$\xi_{0.49} = \boxed{\mu - 0.025\,\sigma}.$$