1) For $x$ belonging to the support, $X$ has cdf

$$F_X(x) = P(X \leq x) = \int_0^x t^{-1/2}/4 \, dt = \sqrt{x}/2.$$  

The inverse function of this is $4x^2$, and so the desired function of $U$ is $4U^2$.

2) Setting the expressions for the mean and variance given on p. 219 of the text equal to the given values for these moments gives us two equations in two unknowns that can be solved to yield the values of the two parameters. In the book's notation we have $a = 3$ and $b = 3$, and so the pdf of $X$ is

$$\frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} x^2(1-x)^2 I_{(0,1)}(x),$$

and so for the desired expectation we have

$$E(1/X) = \int_0^1 x^{-1} \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} x^2(1-x)^2 \, dx$$

$$= \frac{\Gamma(6)\Gamma(2)}{\Gamma(5)\Gamma(3)} \int_0^1 \frac{\Gamma(5)}{\Gamma(2)\Gamma(3)} x(1-x)^2 \, dx$$

$$= \frac{(5!)(1!)}{(4!)(2!)} = 5/2$$

(where the last integral equals 1 because it’s the integral of a pdf over the support of the distribution). Alternatively, one can note that the pdf equals

$$30(x^2 - 2x^3 + x^4) I_{(0,1)}(x),$$

and obtain the desired expectation by multiplying by $x^{-1}$ and integrating, giving us

$$E(1/X) = \int_0^1 30(x - 2x^2 + x^3) \, dx = 30(1/2 - 2/3 + 1/4) = 5/2.$$  

3) Noting that we have

$$F_X(x) = P(X \leq x) = \Phi([x - \mu_X]/\sigma_X) = \Phi([x - 1]/3),$$

for the desired probability we have

$$P(X^2 \leq 4) = P(-2 \leq X \leq 2) = P(X \leq 2) - P(X \leq -2) = \Phi([2 - 1]/3) - \Phi([-2 - 1]/3)$$

which is approximately equal to

$$\Phi(0.33333) - \Phi(-1) \approx 0.6306 - 0.1587 \approx 0.472.$$  

(Note: Linear interpolation was used to obtain $\Phi(0.3333)$. We have

$$\Phi(1/3) \approx \Phi(0.3333) \equiv 0.667\Phi(0.33) + 0.333\Phi(0.34) \equiv 0.667(0.6293) + 0.333(0.6331) \approx 0.6306,$$
but the value for the final answer should be rounded to the nearest thousandth to avoid expressing too much
accuracy when using an approximate method (and since the values given in the standard normal cdf table
have already been rounded).

4) Letting $Y$ be the number of “doubles” in 64 flips, $Y$ has a binomial $(64, 1/6)$ distribution, and the
probability of interest is
$$P(Y \geq 15) = 1 - P(Y \leq 14).$$

For the desired approximation we have
$$1 - \Phi([14 + 1/2 - 64(1/6)]/\sqrt{64(1/6)(5/6)}) = 1 - \Phi(1.28574) \approx 0.099.$$

(Note: Summing the actual binomial distribution probabilities gives us a value of about 0.103, so in this case
the approximation is accurate to only two significant digits.)

5)
(a) Below I’ll list all of the possible outcomes, their probabilities, and the corresponding values of $X$ and
$Y$. The outcomes are represented by ordered pairs of the form $(d_1, d_2)$, where $d_i$ is the result of die $i$.

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>1/16</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(1,2)</td>
<td>1/16</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(1,3)</td>
<td>1/16</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(1,4)</td>
<td>1/16</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(2,1)</td>
<td>1/16</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(2,2)</td>
<td>1/16</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(2,3)</td>
<td>1/16</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(2,4)</td>
<td>1/16</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(3,1)</td>
<td>1/16</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(3,2)</td>
<td>1/16</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(3,3)</td>
<td>1/16</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(3,4)</td>
<td>1/16</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>(4,1)</td>
<td>1/16</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(4,2)</td>
<td>1/16</td>
<td>1</td>
<td>4</td>
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<tr>
<td>(4,3)</td>
<td>1/16</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>(4,4)</td>
<td>1/16</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Below is the joint pmf of $X$ and $Y$ in tabular format.

<table>
<thead>
<tr>
<th></th>
<th>$y = 1$</th>
<th>$y = 2$</th>
<th>$y = 3$</th>
<th>$y = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>1/16</td>
<td>3/16</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x = 1$</td>
<td>0</td>
<td>0</td>
<td>4/16</td>
<td>4/16</td>
</tr>
<tr>
<td>$x = 2$</td>
<td>0</td>
<td>0</td>
<td>1/16</td>
<td>3/16</td>
</tr>
</tbody>
</table>

(b) The marginal pmf for $Y$ can be obtained by summing the probabilities in the columns of the table
above. The result of this is
$$p_Y(y) = \begin{cases} 
7/16, & y = 4, \\
5/16, & y = 3, \\
3/16, & y = 2, \\
1/16, & y = 1, \\
0, & \text{otherwise.}
\end{cases}$$

6)
(a) The support of $X$ is $(0, \infty)$, and so $f_X(x) = 0$ for $x \leq 0$. For $x > 0$,

$$f_X(x) = \int_0^x 2e^{-x}e^{-y} \, dy = 2e^{-x}(-e^{-y}) \bigg|_0^x = 2e^{-x}(1 - e^{-x}) = 2e^{-x} - 2e^{-2x}.$$
Altogether we have
\[ f_X(x) = (2e^{-x} - 2e^{-2x}) I_{(0, \infty)}(x). \]

(b) No, they are no independent. The product of the marginal densities is positive in the first quadrant of the \(x\)-\(y\) plane, and so it’s not equal to the joint pdf of \(X\) and \(Y\) (which is not positive in the entire first quadrant of the \(x\)-\(y\) plane).

(c) We have
\[
P(Y < X/2) = \int_0^{\infty} \int_0^{\infty/2} 2e^{-x}e^{-y} \, dy \, dx
= \int_0^{\infty} 2e^{-x}(-e^{-y}|_{0}^{\infty/2}) \, dx
= \int_0^{\infty} 2e^{-x}(1 - e^{-x/2}) \, dx
= \int_0^{\infty} 2e^{-x} \, dx - \int_0^{\infty} 2e^{-(3/2)x} \, dx
= 2 \int_0^{\infty} e^{-x} \, dx - (4/3) \int_0^{\infty/2} e^{-(3/2)x} \, dx
= 2 - 4/3
= 2/3
\]
(where the last two integrals above both equal 1 because they are integrals of exponential random variable pdfs over the support of the random variables).

(d) The support of \(V\) is \((0, \infty)\), and so \(f_V(v) = 0\) for \(v \leq 0\) (values not belonging to the support). For \(v \in (0, \infty)\),
\[
F_V(v) = P(X + Y \leq v)
= \int_0^{\infty/2} \int_y^{v-y} 2e^{-x}e^{-y} \, dx \, dy
= \int_0^{\infty/2} 2e^{-y}(-e^{-x}|_{y}^{v-y}) \, dy
= \int_0^{\infty/2} 2e^{-y}(e^{-y} - e^{-(v-y)}) \, dy
= \int_0^{\infty/2} 2e^{2y} \, dy - 2e^{-v} \int_0^{\infty/2} \, dy
= (-e^{-y}|_{0}^{\infty/2}) - 2e^{-v}(y|_{0}^{\infty/2})
= 1 - e^{-v} - 2e^{-v}(v/2)
= 1 - e^{-v} - ve^{-v},
\]
and
\[
f_V(v) = \frac{d}{dv}(1 - e^{-v} - ve^{-v}) = 0 + e^{-v} - e^{-v} + ve^{-v} = ve^{-v}.
\]
Altogether, we have
\[
f_V(v) = ve^{-v} I_{(0, \infty)}(v).
\]