Solutions for HW 6
STAT 544, Spring 2015

1) (a) Upon differentiating the pdf we obtain
\[ f_X(x) = \frac{1}{x^2} e^{-1/x} I_{(0, \infty)}(x). \]

(b) We can set \( F_X(\eta X) \) equal to \( 1/2 \) and solve for \( \eta_X \). Clearly, \( \eta_X \) must be some value in the support of \( X \), and so we have
\[ e^{-\eta X} = \frac{1}{2}, \]

which implies that
\[ \eta_X = \frac{1}{\log 2} \approx 1.4427. \]

2) (a) A sketch of the function \( g(x) = x(2-x) \) on the interval \((0, 2)\) indicates that the support of \( Y \) is \((0, 1]\) (since \( X(2-X) \) will assume a value in \((0, 1]\) if \( X \) assumes a value in \((0, 2)\)). So for \( y \leq 0 \), we clearly must have \( F_Y(y) = 0 \) (since it’s impossible for \( Y \) to assume any value below it’s support). For \( y \geq 1 \), \( F_Y(y) = 1 \) (since \( Y \) will always assume a value less than or equal to any upper bound of it’s support).

For \( y \in (0, 1] \),
\[ F_Y(y) = P(Y \leq y) = P(X(2-X) \leq y) = P(-X^2 + 2X - y \leq 0). \]
The solutions to the quadratic equation \(-x^2 + 2x - y = 0\) are \( 1 \pm \sqrt{1 - y} \), which belong to the support of \( X \) for \( y \) belonging to \((0, 1]\). Now, upon looking at a sketch of the function \( g(x) = x(2-x) \) on the interval \((0, 2)\), it can be determined that \( g(x) \) increases on \((0, 1]\), decreases on \((1, 2)\), takes values in \((y, 1]\) on \((1 - \sqrt{1-y}, 1 + \sqrt{1-y}, 2)\), and takes values less than or equal to \( y \) on \((0, 1 - \sqrt{1-y}] \cup [1 + \sqrt{1-y}, 2)\).

Hence, an event equivalent to \( \{ -X^2 + 2X - y \geq 0 \} \), and thus also equivalent to \( \{ Y \leq y \} \), is \( \{ X \leq 1 - \sqrt{1-y} \} \cup \{ X \geq 1 + \sqrt{1-y} \} \). It follows that for \( 0 < y < 1 \),
\[ F_Y(y) = P(Y \leq y) = P(X \leq 1 - \sqrt{1-y}) + P(X \geq 1 + \sqrt{1-y}) = \int_0^{1-\sqrt{1-y}} x/2 \, dx + \int_{1+\sqrt{1-y}}^{\infty} x/2 \, dx = \frac{x^2}{4} \bigg|_0^{1-\sqrt{1-y}} \ + \frac{x^2}{4} \bigg|_{1+\sqrt{1-y}}^{\infty} = 1 - \sqrt{1-y}. \]

(Note that this function increases from 0 to 1 as \( y \) increases from 0 to 1 (corresponding to the support of \( Y \), which should be the case for the cdf of this continuous random variable.) Altogether, we have
\[ F_Y(y) = \begin{cases} 1, & y \geq 1, \\ 1 - \sqrt{1-y}, & 0 < y < 1, \\ 0, & y \leq 0. \end{cases} \]

(b) Using the pdf of \( X \), we have
\[ E(X(2-X)) = E(2X - X^2) = \int_{-\infty}^{\infty} (2x - x^2) f_X(x) \, dx = \int_0^2 (2x - x^3/2) \, dx = (x^2/3 - x^4/8)|_0^2 = 2/3. \]
(Alternatively, the pdf of $Y$ can be used, but the integration isn’t as simple.)

(c) We have

\[ E(X) = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{0}^{2} x (x/2) \, dx = \int_{0}^{2} x^2/2 \, dx = x^3/6 \bigg|_{0}^{2} = 4/3 \]

and

\[ E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_{0}^{2} x^2 (x/2) \, dx = \int_{0}^{2} x^3/2 \, dx = x^4/8 \bigg|_{0}^{2} = 2. \]

So

\[ \text{Var}(X) = E(X^2) - [E(X)]^2 = 2 - (4/3)^2 = 2/9. \]