STAT 652: HW #5

Spring 2020

Instructions: Please present your solutions in order. As always, use paper which is approximately 8.5 inches by 11 inches, present tidy and easy-to-follow solutions, *draw boxes around your final answers* (but don't just give answers without supporting work), and staple all sheets together in the upper left hand corner. I don't like cover sheets, executive summaries, folders, binders, or paper clips.

1) $X_1, X_2, \ldots, X_n$ are iid random variables having pmf

$$f_X(x|\theta) = \theta^2(1-\theta)^x(1+x)I_{\{0,1,2,\ldots\}}(x),$$

where $\theta \in \Theta = (0,1)$. Consider a prior distribution for θ having pdf

$$\pi(\theta) = 1092\,\theta^2(1-\theta)^{11}\,I_{(0,\,1)}(\theta).$$

(a) (4 points) Given an observed sample $x_1, x_2, \ldots, x_n$ from the random variables introduced above, and the prior density given above, determine the posterior density for θ.

(b) (4 points) Using the posterior density requested in part (a), give the Bayes estimate of θ based on the squared-error loss function.

(c) (8 points) Now consider the loss function

$$l(\theta,\hat{\theta}) = \left(\frac{\hat{\theta}-\theta}{\theta}\right)^2$$

and use it to determine the Bayes estimate based on the x_i and the given prior distribution density. (*Note*: With this loss function we do not have that the Bayes estimate of θ is $E(\theta|\vec{X} = \vec{x})$. Instead, the Bayes estimate is the value of b which minimizes

$$\int_\Theta l(\theta,b)\pi(\theta|\vec{x})\,d\theta \;=\; b^2\int_\Theta \frac{1}{\theta^2}\pi(\theta|\vec{x})\,d\theta - 2b\int_\Theta \frac{1}{\theta}\pi(\theta|\vec{x})\,d\theta + \int_\Theta \pi(\theta|\vec{x})\,d\theta.$$

The right side of the equation above is really a fairly *simple* function of b (with messy looking coefficients, but each is an integral that shouldn't be too hard to evaluate, given the resemblance of the integrands to beta distribution densities), and it can be used to develop an expression for the minimum by taking a derivative. (*Note*: You are to evaluate the integrals using the posterior density from part (a).))

2) Consider iid random variables $X_1, X_2, \ldots, X_n$ having pdf

$$f_X(x|\theta) = \begin{cases} \frac{2x}{\theta^2}, & 0 < x \le \theta, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta \in \Theta = (0,\infty)$.

(a) (8 points) Give the Pitman estimator, $\hat{\theta}_P$, of θ.

(b) (4 points) Give the MSE of the estimator requested in part (a).

(c) (4 points) Give the MLE of θ.

(d) (4 points) Give the MSE of the estimator requested in part (c). (*Note*: You should find that the expected loss requested for this part is about twice as large as the expected loss requested in part (b). (The limit of the ratio of the two risks is 2, with the Pitman estimator being the one having the smaller risk.) If one found the MME based on the first sample moment, and determined its MSE, it could be seen that it can be much larger than the MSEs of the other two estimators considered here. However, the UMVUE is pretty good in this situation; its MSE is only slightly larger than the MSE of the Pitman estimator.)