1) For HW 4 it was determined that $E(X) = 49/3$. In a similar manner we can obtain the 2nd moment of $X$:

$$E(X^2) = \sum_{x=1}^{24} x^2 \left( \frac{x}{300} \right) = \frac{1}{300} \sum_{x=1}^{24} x^3 = \frac{1}{300} \left( \frac{24(25)}{4} \right)^2 = 300.$$ 

It follows that we have 

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 300 - \left( \frac{49}{3} \right)^2 = \frac{299}{9} = 33.222.$$ 

2) The desired 3rd moment is 

$$\sum_{x=11}^{20} x^3 \left( \frac{1}{10} \right) = \frac{1}{10} \left( \sum_{x=1}^{20} x^3 - \sum_{x=1}^{10} x^3 \right) = \frac{1}{10} \left( \frac{20(21)^2}{4} - \frac{10(11)^2}{4} \right) = (44100 - 3025)/10 = 4107.5.$$ 

3) (a) For $X$ to equal $x$, we must either have $x - 2$ alternating outcomes of Heads and Tails, ending in Heads, followed by 2 Tails or $x - 2$ alternating outcomes of Heads and Tails, ending in Tails, followed by 2 Heads. Each of these sequences of coin flip outcomes has probability $(1/2)^x$, and so the overall probability that $X = x$ is $2(1/2)^x = 1/2^{x-1}$. Noting that the possible values of $X$ are 2, 3, 4, ..., for the pmf of $X$ we have 

$$p_X(x) = \frac{1}{2^{x-1}} I_{\{2,3,4,...\}}(x).$$ 

(b) Note for for the mean of a geometric (0.5) random variable we have 

$$2 = 1/(1/2) = \sum_{x=1}^{\infty} x(1/2)^x = 1(1/2) + 2(1/2)^2 + 3(1/2)^3 + 4(1/2)^4 \cdots.$$ 

If we multiply both sides by 2 we obtain 

$$4 = 1 + 2(1/2) + 3(1/2)^2 + 4(1/2)^3 \cdots,$$ 

and so it follows that 

$$3 = 2(1/2) + 3(1/2)^2 + 4(1/2)^3 \cdots.$$ 

But taking notice of the pmf obtained for part (a), the sum on the right (just above) is an expression for $E(X)$, and so we have that $E(X) = 3$. (Note: With a geometric (0.5) random variable, there is a 0.5 chance of success on each trial, starting from the 1st trial, and the expected value equals 2. With our random variable $X$ there is a 0.5 chance of success on each trial, from the 2nd trial onwards (since from trial 2 onwards, each time the probability is 0.5 that the outcome will be the same as the immediately preceding outcome), and so it makes sense that the mean of $X$ is larger than the mean of a geometric (0.5) random variable by 1. Alternatively, one can note that the probability that $X$ assumes the value 2 is the same as the probability that a geometric (0.5) random value assumes the value 1, the probability that $X$ assumes the value 3 is the same as the probability that a geometric (0.5) random value assumes the value 2, the probability that $X$ assumes the value 4 is the same as the probability that a geometric (0.5) random value assumes the value 3, and so on. Thus the probability mass of $X$’s distribution is distributed the same way as the probability mass of a geometric (0.5) distribution, except that it is shifted up by 1, and so it makes sense that the mean of $X$ is larger by the value 1.) 

4) (a) Letting $N(2)$ be the number of major accidents that will occur during the 2 week period, $N(2)$ is a Poisson random variable having mean 

$$\frac{2.2}{\text{week}} \times 2 \text{ week} = 4.4.$$
and the desired probability is

\[ P(N(2) \geq 3) = 1 - P(N(2) \leq 2) \]
\[ = 1 - [P(N(2) = 0) + P(N(2) = 1) + P(N(2) = 2)] \]
\[ = 1 - [e^{-4.4} + 4.4e^{-4.4} + (4.4)^2e^{-4.4}/2!] \]
\[ \approx 0.815. \]

(b) Letting \( N(1) \) be the number of major accidents that will occur during a 1 week period, \( N(1) \) is a Poisson random variable having mean

\[ \frac{2.2}{\text{week}} \times 1 \text{ week} = 2.2. \]

Letting \( p \) denote \( P(N(1) = 0) = e^{-2.2} \approx 0.110803 \), the desired expectation is that of a binomial \((12, p)\) random variable, which is

\[ 12p = 12e^{-2.2} = 3.330. \]

5) Letting \( Y \) be the number of white balls obtained when 3 balls are randomly drawn from such an urn, \( Y \) has a hypergeometric distribution. Letting \( p \) denote the probability of obtaining more white balls than black balls, we have

\[ p = P(Y \geq 2) = \frac{\binom{6}{2}\binom{4}{1}}{\binom{10}{3}} + \frac{\binom{6}{3}\binom{4}{0}}{\binom{10}{3}} = \frac{15(4) + 20(1)}{120} = \frac{80}{120} = 2/3. \]

Letting \( X \) be the number of times more white balls than black balls are obtained, \( X \) is a binomial \((5, p)\) random variable, and we have that

\[ E(X) = 5 \left( \frac{2}{3} \right) = \frac{10}{3} \approx 3.333. \]

6) (a) Below the support, the cdf equals 0, and so \( F_X(x) = 0 \) for \( x \leq 0 \). Above the support, the cdf equals 1, and so \( F_X(x) = 1 \) for \( x \geq 2 \). For values belonging to the support \((x \in (0, 2))\), we can integrate the cdf to obtain

\[ F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t) \, dt = \int_{0}^{x} \left( \frac{3}{2}t^2 - \frac{3}{4}t^3 \right) \, dt = \frac{x^3}{2} - \frac{3}{16}x^4. \]

Altogether, we have

\[ F_X(x) = \begin{cases} 
 1, & x \geq 2, \\
 x^3/2 - 3x^4/16, & 0 < x < 2, \\
 0, & x \leq 0.
\end{cases} \]

(b) Using the cdf, for the desired probability we have

\[ P(X > 3/2) = 1 - P(X \leq 3/2) = 1 - F_X(3/2) = 1 - (27/16 - 243/256) = 67/256. \]

(One can also obtain this value using \( \int_{3/2}^{\infty} f_X(x) \, dx \).)

(c) We have that

\[ E(X) = \int_{-\infty}^{\infty} x f_X(x) \, dx \]
\[ = \int_{0}^{2} x \left( \frac{3}{2}x^2/2 - \frac{3}{4}x^3/4 \right) \, dx \]
\[ = \int_{0}^{2} \left( \frac{3}{2}x^3/2 - \frac{3}{4}x^4/4 \right) \, dx \]
\[ = \left( \frac{3x^4}{8} - \frac{3x^5}{20} \right) \bigg|_{0}^{2} \]
\[ = 6/5. \]