1) $X_1, X_2, \ldots, X_n$ are iid random variables having pdf

$$f_X(x|\theta) = \frac{e^{-x/\theta}}{\sqrt{\theta \pi x}} I_{(0, \infty)}(x),$$

where $\theta > 0$.

(a) (2 points) For estimating $\theta$, the MLE, UMVUE, and MME based on the first sample moment are all equal to $2X$. Give the MSE of this estimator.

(b) (8 points) Give the Pitman estimator of $\theta$.

(c) (6 points) Give the MSE of the Pitman estimator of $\theta$. (Don’t give the expected modified squared error loss. Give the ordinary expected squared error loss.) (Note: With regard to partial credit, it may be helpful if you give the mean, bias, and variance of the estimator in case you screw up the determination of the MSE.) (Note: You should find that

$$\frac{\text{MSE}(\hat{\theta}_{Pit})}{\text{MSE}(\hat{\theta}_{UMVUE})}$$

does not depend on $\theta$, is less than 1, and that the limit of this ratio as $n$ tends to infinity is 1. So for small samples the Pitman estimator may be appreciably better, but it makes little difference for larger samples.)

2) (5 extra credit points) Let $X_1, X_2, \ldots, X_n$ be iid random variables having pdf

$$f_X(x) = \sqrt{2/\pi} \exp\left(-\frac{(x - \theta)^2}{2}\right) I_{(\theta, \infty)}(x),$$

where $\theta \in (-\infty, \infty)$. Give the Pitman estimate of $\theta$ based on the sample $x_1, x_2, \ldots, x_n$. (If you evaluate your expression for the sample $x_1 = 0.99, x_2 = 0.40, x_3 = 1.48, x_4 = 0.73$, you should obtain an estimate of about 0.14 — but for your answer, give the estimate for arbitrary $x_i$.)

3) Suppose that we observe iid random variables $X_1, X_2, \ldots, X_n$ having pmf

$$p_X(x|\theta) = \theta(1-\theta)^{x-1} I_{\{1,2,3,\ldots\}}(x),$$

where $\theta \in \Theta = (0, 1)$. Consider the prior density $\pi(\theta) = I_{(0,1)}(\theta)$.

(a) (10 points) Find the posterior density, and give the Bayes estimate of $\theta$ based on the squared-error loss function. (Note: Be sure to give both items that are requested — draw a box around (or highlight) each on your submitted solutions.)

The following parts may be useful to you in checking your work for this problem, but they will not be graded. There is no need to submit solutions for these parts.

(b) (0 points) The mode of the posterior density (the value of $\theta$ which maximizes $p(\theta|x)$) is sometimes called the generalized maximum likelihood estimate. (Note that the generalized mle doesn’t depend on the choice of the loss function. We can employ this estimate even if we don’t identify an appropriate loss function. Of course, if you can identify an appropriate loss function, then it makes sense to use it by finding the estimator which minimizes the Bayes risk. In part (a), we consider a squared-error loss function, and the corresponding Bayes estimator is the mean of the posterior distribution instead of its mode.) Use the posterior density requested in part (a) and find the generalized mle. (The estimate is $\frac{1}{x}$, which is different from, but similar to, the estimate from part (a) — for large sample sizes the two estimates will be very close to one another.)
(c) (0 points) Show that the Bayes estimator corresponding to part (a) is a consistent estimator.

4) (14 points) Consider the $N(\theta, 1)$ random variable $X$, the prior density $\pi(\theta) = (2\pi)^{-1/2} \exp(-\theta^2/2)$ (for all $\theta \in (-\infty, \infty)$), and the loss function

$$l(\theta, \hat{\theta}) = \exp\left(\frac{\hat{\theta} - \theta}{2}\right) - \frac{\hat{\theta} - \theta}{2} - 1.$$  

(Note: This is a special case of Zellner’s LINEX loss function. It is asymmetric in that it penalizes overestimation differently than underestimation.) Find the posterior density, and give the Bayes estimate based on the loss function given above. (Note: Be sure to give both items that are requested — draw a box around (or highlight) each on your submitted solutions.) Note that the estimate is to be based on a single observation, $x$, and not a sample of size $n$. (Note: With this loss function we do not have that the Bayes estimate of $\theta$ is $E(\theta | X = x)$. Instead, the Bayes estimate is the value of $a$ which minimizes

$$\int_{\Theta} l(\theta, \hat{\theta}) p(\theta | x) \, d\theta = \int_{\Theta} e^{(a-\theta)/2} p(\theta | x) \, d\theta - \frac{1}{2} a \int_{\Theta} p(\theta | x) \, d\theta + \frac{1}{2} \int_{\Theta} \theta p(\theta | x) \, d\theta - \int_{\Theta} p(\theta | x) \, d\theta.$$  

While evaluating the first of these integrals may be a bit messy, the other ones should be relatively easy if you keep in mind that the $\pi(\theta | x)$ is a valid pdf.)