STAT 652: HW #4

Spring 2020

Instructions: As usual, *draw boxes around your final answers, and don't just give answers without adequate justification.*

You can make use of results given in the class notes or the text — it's not necessary to derive results from the notes in your solution as long as you clearly indicate the page number(s) that justify your claims. (That is, to avoid showing justification for something derived in the notes or text, you can just cite the proper pages.) For example, if the notes (or the text) establishes/gives that a certain statistic is a complete sufficient statistic, you don't have to justify that it is if you make reference to a particular page from the notes or text. But for the general result about certain statistics pertaining to exponential families being complete sufficient statistics, you should show that you're dealing with an exponential familiy and that any needed conditions are satisfied for the case at hand. For this assignment you may choose to justify that a given estimator is a UMVUE by claiming that it's a function of a complete sufficient statistic that is unbiased for the indicated estimand. But if you do this, be sure to justify that your estimator is indeed unbiased for the estimand (maybe by appealing to the Rao-Blackwell theorem in at least one case), and you should also indicate why a certain statistic is a complete sufficient statistic. (Reviewing examples from the class notes may be a good idea.) (In general, you should also show that your estimator has finite variance, but for this assignment you're to obtain the variances of the two UMVUEs requested in parts (a) and (c) of Problem 1 in other parts of that problem (and so there's no need to also obtain the variances in your solutions for parts (a) and (c)). The UMVUE requested in Problem 2 is an estimator of a probability. Since the estimand is a value between 0 and 1, and the UMVUE is a good estimator of the estimand, one can guess that it won't assume extreme values with probabilities large enough to make the variance infinite (and so I won't request that you obtain the exact variance). Also, since the variance of the estimator requested in Problem 3 is rather messy to obtain, I'll let you skip doing that step.)

1) Let $X_1, X_2, \ldots, X_n$ be iid random variables having pdf

$$f_X(x|\theta) = \theta\,(1+x)^{-(1+\theta)}\,I_{(0,\infty)}(x),$$

where $\theta > 0$.
   (a) (6 points) Give the UMVUE of $\frac{1}{\theta}$.
   (b) (4 points) Give the variance of the UMVUE of $\frac{1}{\theta}$.
   (c) (6 points) Give the UMVUE of θ.
   (d) (4 points) Give the CRLB for unbiased estimators of θ.
   (e) (4 points) Give the variance of the UMVUE of θ.

2) (8 points) $X_1, X_2, \ldots, X_n$ are iid Poisson(θ) random variables, where $\theta \in \Theta = (0\,\infty)$. Give the UMVUE of $\theta e^{-\theta}$ $(= P(X_i = 1))$.

3) $X_1, X_2, \ldots, X_n$ are iid N(μ, σ^2) random variables.
   (a) (8 points) For the case of $n = 4$, give the UMVUE of μ^2/σ. (You may want to first work with general n, but be sure to simplify your result to give a fairly simple estimator for the particular case of $n = 4$.) Express the desired UMVUE in terms of $\overline{X}$, S, integers, and π. (You can have a square root or two, but don't express it using any gamma functions.) (*Hints*: You may desire to use the fact that for a sample of iid normal random variables, the sample mean and the sample variance are stochastically independent. You may also desire to use the fact that for iid N(μ, σ^2) random variables, $\sigma^{-2}\sum_{i=1}^n (X_i - \overline{X})^2 \sim \chi^2_{n-1}$. Using this last fact, you might start an attack on this problem by finding $E(1/S)$.)
   (b) (2 points) Give the lower bound for the variance of unbiased estimators of μ^2/σ obtained from the information inequality. (You can make use of the information matrix given in the class notes (on p. 7.3.39). It's important to note that if you use the information on p. 7.3.39 of the class notes, you need to treat the parameters as μ and σ^2, not μ and σ. (I.e., when obtaining the gradient, you need to treat σ^2 as the parameter. If you treat σ as the parameter when obtaining the gradient vector, you'd need to use an information matrix based on treating σ as the parameter (instead of σ^2).))