STAT 652: HW #3

Spring 2020

Instructions: Present well-organized and neat solutions. (Please present your solutions in order. For example, your solution to Problem 2 should precede your solution to Problem 3, and if a problem has two parts, part (a) should come before part (b). As always, use paper which is approximately 8.5 inches by 11 inches, present tidy and easy-to-follow solutions, draw boxes around or highlight your final answers (but don't just give answers without supporting work), and staple all sheets together in the upper left hand corner. I don't like cover sheets, executive summaries, folders, binders, or paper clips.)

Note: If I request a numerical estimate, give either the exact value, or a value which has been rounded to three significant digits. (23.6, 0.236, and 0.00236 all have three significant digits. 0.024 only has two significant digits.)

Note: For maximum likelihood estimates/estimators, be sure to clearly establish that you've obtained a maximizing value. (E.g., if you set the derivative of the likelihood function equal to 0, be sure to establish that the solution is a maximizing value and not a minimum or a point of inflection, and additionally argue that you've found the global maximum and not just a local maximum.) Also, if I request an estimate or estimator which doesn't exist, you should *fully explain* why the estimate or estimator cannot be given.

Note: Below I use MLE to denote maximum likelihood *estimator* and mle to denote maximum likelihood *estimate*. Similarly, I use MME to denote method of moments *estimator* and mme to denote method of moments *estimate*. Estimators should be expressed using upper-case (e.g., X_i), and estimates should be expressed using lower-case (e.g., x_i). When writing likelihood functions, lower-case should be used (e.g., x_i). (Be sure to write clear enough so that X_i can be distinguished from x_i.)

1) Let $X_1, X_2, \ldots, X_n$ be iid random variables having pdf

$$f(x|\theta) = \frac{\theta x^{\theta-1}}{3^\theta} \, I_{(0,\,3)}(x),$$

where $\theta > 0$.

(a) (4 points) Give a MME of θ based on the first sample moment. (As a way to check your answer, the MME can easily be shown to be a consistent estimator using the WLLN.)

(b) (6 points) Give the MLE of θ.

(c) (2 points) Use the WLLN (and perhaps some other results that you used on HW #1 and HW #2) to show that the MLE converges in probability to θ (i.e., that the estimator is consistent).

2) Let $X_1, X_2, \ldots, X_n$ be iid random variables having pdf

$$f(x|\theta) = \frac{\theta + 1 - x}{2} \, I_{[\theta-1,\,\theta+1)}(x),$$

where $\theta \in \Theta = (-\infty,\, \infty)$.

(a) (4 points) Give a MME of θ based on the first sample moment.

(b) (6 points) Give the MLE of θ.

3) (6 points) $U_1, U_2, \ldots, U_n$ are iid random variables having pdf

$$f(u) = (2\delta)^{-1} \, I_{[\gamma-\delta,\,\gamma+\delta]}(u).$$

Give method of moments estimates of γ and δ (rounded to the nearest hundredth) for the case of $n = 3$, $u_1 = 1.73$, $u_2 = -0.77$ and $u_3 = 0.59$.

4) Suppose that for births of a certain species we have:

$P(\text{color-blind male}) = \frac{\theta}{2}$,
$P(\text{non-color-blind male}) = \frac{(1-\theta)}{2}$,
$P(\text{color-blind female}) = \frac{\theta^2}{2}$,
$P(\text{non-color-blind female}) = \frac{(1-\theta^2)}{2}$,

where $0 < \theta < 1$. Assume that 200 independent observations from this distribution results in a sample consisting of 100 males, 11 of whom are color-blind, and 100 females, 2 of whom are color-blind.

(a) (6 points) Based on this sample, give an estimate of θ using the method of maximum likelihood. (Give a numerical value.)

(b) (4 points) Based on the sample, give an estimate of θ using a frequency substitution estimator which is unbiased. (You do not have to explicitly establish the unbiasedness (although it isn't hard to do so). Just indicate the form of the estimator and give the corresponding numerical estimate. Note that it's possible to obtain a frequency substitution estimator which is biased, so be sure to obtain your estimate using one that's unbiased.)

5) $X_1, X_2, \ldots, X_n$ are iid random variables having pdf

$$f(x|\theta) = \frac{(1 + \theta x)}{2} I_{[-1,1]}(x),$$

where $\theta \in \Theta = [-1, 1]$.

(a) (4 points) Give the MME of θ based on the first sample moment.

(b) (6 points) Now consider the case of $n = 1$, and give a formula for the mle of θ which results from a single observed value, x. (You should find that the mle isn't unique if $x = 0$ — more than one value of $\theta \in \Theta$ maximizes the likelihood.)

(c) (6 points) Now consider the case of $n = 4$, and particular sample of values $x_1 = 0.5$, $x_2 = -0.1$, $x_3 = 0.9$, and $x_4 = -0.5$, and give a numerical value for the mle of θ, rounding to the nearest thousandth. (I suggest that you work with the likelihood, as opposed to the log-likelihood, for this problem. If you plug in the values for the x_i and multiply the factors, you'll get a fourth degree polynomial for the likelihood. To maximize this, you can set its derivative to 0 and obtain a root of the equation utilizing Newton's method, using the mme (obtained by plugging the x_i into your part (a) result) as an initial value. If you do this (correctly), you'll converge to the mle in just a few iterations. (It would be nice if you also plotted the likelihood function in order to see that your solution is indeed the maximizing value of $\theta \in \Theta$, but for this problem I won't require that you show work to firmly establish that your estimate is the unique maximizing value.) Alternatively, you may choose to use some sort of software to obtain the mle (but if you do that, be sure to provide me with some sort of justification for your final answer (e.g., show some output from the software).)