STAT 652: HW #2

due Feb. 28, 2012

- To maximize partial credit, present carefully organized and neat solutions, explaining your approach and showing adequate justification for your answers.
- Please present your solutions in order. For example, your solution to Problem 1 should precede your solution to Problem 2, and your solution to Problem 2 should precede your solution to Problem 3. (Similarly, your solution to part (a) of a problem should come before your solution for part (b).) Use paper which is approximately 8.5 inches by 11 inches, *draw boxes around or highlight* your final answers (but don't just give answers without supporting work), and **staple** all sheets together in the upper *left* hand corner. *I don't like cover sheets, executive summaries, folders, binders, or paper clips.*
- If you use *Maple* or *Mathematica* (or something similar), supply pertinent output. (Don't give me an "appendix" with all of your output in it. Rather, insert each computation into your homework solutions in the appropriate place.)
- While it's okay to discuss problems with other students, you should not copy anyone's work. Also, don't expect me to tell you how to solve these problems. I'll be happy to discuss similar problems, but I want you to figure out how to solve these on your own.
- For problems 2 through 5, *in one or more cases a limiting distribution may not exist*, and in such cases you cannot supply me with what I'm requesting. In such cases you should state that a limiting distribution does not exist and justify your conclusion.

Hints: You *may* desire to use the following facts on one or more of the problems.

(i) If $Z \sim$ N(0,1), then $Z^2 \sim \chi^2_1$. Similarly, recall that if X is an exponential random variable and $\alpha > 0$, then X^α has a Weibull distribution (see p. 102 of text). (What does this suggest about a Weibull random variable raised to a positive power?)

(ii) If $|x| < 1$, then $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$.

(iii) $\sum_{k=1}^n k = \frac{n(n+1)}{2}$.

1) (4 points) $T_1, T_2, \ldots$ are random variables for which $T_n \sim \text{Normal}\,(0,\, 1/n^2)$. Show that the sequence $T_1, T_2, \ldots$ converges in probability to 0.

2) (8 points) $T_1, T_2, \ldots$ are random variables for which $T_n \sim \text{Normal}\,(0,\, n^2)$. Consider the sequence $T_1, T_2, \ldots$ and give the pmf or the pdf of the limiting distribution.

3) $X_1, X_2, \ldots$ are iid exponential random variables having pdf

$$f(x) = e^{-x}\, I_{(0,\infty)}(x),$$

and

$$T_n = \sqrt{n}\left(\frac{\overline{X} - 1}{\overline{X}}\right).$$

(a) (8 points) Consider the sequence $T_1, T_2, \ldots$ and give the pmf or pdf of the limiting distribution.

(b) (4 points) Letting $V_n = T_n^2$, consider the sequence $V_1, V_2, \ldots$ and give the pmf or pdf of the limiting distribution.

4) (8 points) $X_1, X_2, \ldots$ are mutually independent random variables for which $X_k \sim \chi^2_k$, and

$$T_n = \frac{2\sum_{k=1}^n X_k}{n} - n.$$

(*Note*: The X_k are not identically distributed. They have different chi-squared distributions. E.g., $X_1 \sim \chi^2_1$ and $X_2 \sim \chi^2_2$.) Consider the sequence $T_1, T_2, \ldots$ and give the pmf or the pdf of the limiting distribution.

5) (8 points) $X_1, X_2, \ldots$ are iid random variables having pdf

$$f_X(x) = \frac{2}{x^3}\, I_{(1,\infty)},$$

and

$$T_n = \max\{X_1, \ldots, X_n\}/\sqrt{n}.$$

Consider the sequence $T_1, T_2, \ldots$ and give the pmf or pdf of the limiting distribution.

6) $X_1, X_2, \ldots, X_n$ are iid random variables having pdf

$$f_X(x|\theta) = \frac{1}{\theta} e^{-x/\theta} \, I_{(0,\infty)}(x).$$

$1/\overline{X}_n^2$ can be used to estimate $1/\mu_X^2 = 1/\theta^2$.

(a) (2 points) Give, in terms of θ, the first-order approximation of the mean of $1/\overline{X}_n^2$.

(b) (4 points) Give, in terms of θ, the second-order approximation of the mean of $1/\overline{X}_n^2$.

(c) (4 points) Give, in terms of θ, the first-order approximation of the variance of $1/\overline{X}_n^2$.

(d) (2 points) Give the values of a and b, with $a > 0$, for which

$$\sqrt{n} \, \frac{\overline{X}_n^{-2} - \theta^{-2}}{a\theta^b}$$

converges in distribution to a standard normal random variable.

7) (4 points) $X_1, X_2, \ldots, X_n$, where $n \geq 2$, are iid random variables having pdf

$$f_X(x|\theta) = e^{(\theta - x)} \, I_{(\theta,\infty)}(x).$$

Give a one-dimensional sufficient statistic.

Comments

Some of you may find this assignment to be rather difficult. If you get stuck, try:

(i) breaking the problem up into smaller parts;

(ii) exploring more than one method of attack.

Don't forget about mgfs, but you don't have to use them on every problem (and I don't think that you should try to use them on every problem). Often, a simple direct attack is all that is needed. Don't forget about the law of large numbers, the central limit theorem, Slutsky's theorem, the definition of convergence in probability, and other powerful results covered in the text and the class notes. (If you encounter a sum of *iid* random variables, then you may first want to consider using either the law of large numbers or the central limit theorem, but of course some other method of attack may also work, and may even work better. It's just that I would typically consider investigating one of these limit laws first. (Note that I stipulated that the random variables in the sum are iid.))

Feel free to use probability distribution facts given in the text and the notes. (That is, you don't have to derive those probability facts as part of your solutions.) If you use a theorem from the text, give the number (or name) of the theorem.