STAT 652 (take-home) FINAL EXAM
Spring 2020

Note: If you submit hand-written solutions, be sure to clearly distinguish X from x. (X should be about twice as tall as x and be written with straight line segments (whereas x should not be written with straight line segments).) Also, be sure to draw boxes around or highlight your final answers.

1) Consider iid random variables $X_1, X_2, \ldots, X_n$ having pdf

$$f(x|\theta) = \frac{\theta}{x^{\theta+1}} I_{(1,\,\infty)}(x),$$

where $\theta > 1$.

(a) (10 points) Give a method of moments estimator of θ and explain why it is a consistent estimator. (A simple explanation can be given based on Sec. 5.5 of the text.)

(b) (15 points) For the case of $n = 25$ and $\sum_i^{25} \log x_i = 7.6$, give an approximate 95% confidence interval for θ based on the asymptotic normality of the MLE of θ.

(c) (10 points) For the case of $n = 25$ and $\sum_i^{25} \log x_i = 7.6$, use the central limt theorem to obtain an approximate p-value for the UMP test of $H_0 : \theta \leq 2$ vs. $H_1 : \theta > 2$.

(d) (10 points) Give the maximum likelihood estimator of the distribution mean.

2) (12.5 points) X_1 and X_2 are iid random variables having pdf

$$f(x|\theta) = \begin{cases} \frac{4}{\theta} - \frac{4x}{\theta^2}, & \theta/2 < x < \theta, \\ \frac{4x}{\theta^2}, & 0 < x \leq \theta/2, \\ 0, & \text{elsewhere.} \end{cases}$$

where $\theta \in \Theta = (0, \infty)$. (*Note*: The density has a triangle shape and is symmetric about $\theta/2$. Making a sketch of the pdf might help you grasp a better understanding of the situation.) For the case of $x_1 = 10.0$, and $x_2 = 4.5$, give the maximum likelihood estimate of θ.

3) (10 points) X_1 and X_2 are iid geometric random variables having pmf

$$f(x|\theta) = \theta(1-\theta)^{x-1} I_{\{1,2,3,\ldots\}}(x),$$

where $\theta \in \Theta = (0, 1)$. Give the unique UMVUE of $f(2|\theta) = \theta(1-\theta)$, being sure to justify your answer.

4) Let X be the number of black balls obtained when a simple random sample of size 4 is drawn (without replacement, in a way such that all possible subsets of size 4 are equally-likely to be selected) from a collection of θ black balls and $12 - \theta$ white balls, where $\theta \in \Theta = \{0, 1, 2, 3\}$. (*Note*: X has a hypergeometric distribution.)

(a) (15 points) Give the test function for a size 0.1 LRT, based on X, of $H_0 : \theta = 2$ vs. $H_1 : \theta \in \{0, 1, 3\}$.

(b) (5 points) What is the power of the test requested in part (a) if $\theta = 1$?

5) Consider 120 independent random variables: $W_1, W_2, \ldots, W_{15}$ are Poisson random variables having mean θ_W, $X_1, X_2, \ldots, X_{30}$ are Poisson random variables having mean θ_X, $Y_1, Y_2, \ldots, Y_{30}$ are Poisson random variables having mean θ_Y, and $Z_1, Z_2, \ldots, Z_{45}$ are Poisson random variables having mean θ_Z.

(a) (12.5 points (plus the possibility of 2 *extra credit points*)) Using asymptotic results pertaining to likelihood ratio tests, determine whether or not a rejection of the null hypothesis occurs if an approximate size 0.05 test is used to test $H_0 : \theta_W = \theta_X = \theta_Y = \theta_Z$ against the general alternative that the means are not all equal, if $\sum_{i=1}^{15} w_i = 60$, and $\sum_{i=1}^{30} x_i = 90$, and $\sum_{i=1}^{30} y_i = 60$, and $\sum_{i=1}^{45} z_i = 90$. You can use the fact that given observations of iid Poisson random variables, the mle of the distribution mean is just the sample mean. (Don't spend time justifying the needed mles.) For 2 *extra credit points* give the approximate p-value.

(b) (3 *extra credit points*) Now consider a test of $H_0 : \theta_X = \theta_Y$ against $H_1 : \theta_X \neq \theta_Y$, but instead of using a likelihood ratio test, base a test on a standardized difference in the sample means, and report an approximate p-value. (Use a test statistic based on $\overline{X} - \overline{Y}$ which by the central limit theorem and Slutsky's theorem has approximately a standard normal distribution when the null hypothesis is true.)