(Take-Home) APPLIED PROBABILITY FINAL EXAM

Spring 2020

Instructions: You can look at whatever books or notes that you desire to use, but *you cannot give help to anyone or receive help from anyone.* You can use a calculator or mathematical software on a computer.

Each of the 10 parts is worth 10 points. For requested probabilities and expected values, give either exact numerical values, or else round to the nearest thousandth; don't give expressions that need to be evaluated for such requested values. *Draw boxes around your answers*, but **to receive full credit you must supply adequate justification for your answers**. (*Note*: For one or more parts of this exam, you might be able to use a clever argument to obtain a correct answer somewhat quickly. That's fine; you can obtain your answers in any legitimate way as long as you indicate how you arrive at your answers.)

1) Suppose that (when large gatherings are once again permissible) 300 people are invited to a party, and suppose that for each person invited, the person will not attend with probability 1/3, the person will attend alone with probability 1/3, and the person will attend and bring one other person as a guest with probability 1/3. Assuming that the actions of the 300 invited people are independent, use the central limit theorem to approximate the probability that at least 320 people attend the party as a result of the 300 invitations.

2) Consider a sequence of independent flips of a biased coin for which the probability of *heads* is 2/3 and the probability of *tails* is 1/3. Letting X be the number of *tails* obtained prior to observing an outcome of *heads*, the pmf of X is

$$p_X(x) = \frac{2}{3^{x+1}} \, I_{\{0,1,2,\ldots\}}(x).$$

Give the mgf of X, $M_X(t)$. You don't have to indicate what values of t are applicable. (Don't express the mgf as an infinite series. Instead, express it as a relatively simple function of t.)

3) X is a uniform $(1/2,\, 5/4)$ random variable (i.e., it's uniformly distributed on the interval $(1/2,\, 5/4)$), Y is a random variable having pdf

$$f_Y(y) = 2y\, I_{(0,\,1)}(y),$$

and X and Y are independent. Give the pdf of $V = \max\{X, Y\}$.

4) Suppose that a subset of 4 balls will be randomly selected and removed from an urn initially containing 6 blue balls and 6 green balls, and then another subset of 4 balls will be randomly selected and removed from the 8 balls that remain (after the first subset of 4 balls is removed). Letting X be the number of these subsets that contain more blue balls than green balls, give the value of $E(X)$. (*Note*: This problem may be solved in more than one way. One way to do it is to let Y_1 be the number of blue balls in the first subset of size 4 that is selected, and use $E(X) = E(E(X|Y_1))$. However, the answer may also be obtained in a *much less messy way* using another technique that I emphasized when covering Ch. 7 in class (and if you obtain the correct answer using this other technique, you can earn up to 3 extra credit points). My advice is to try to solve the problem using at least two different methods as a way to check your answer, but if you do this *only provide me with what you think is your best solution . . . do not submit more than one solution.*)

5) The joint pdf of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-y}, & x > 0, y > 2x, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Give pdf of $W = Y/X$. (*Note*: As a way to check your work, I'll give you that $E(1/W) = 1/4$. But *do not submit any work related to using this check.*)

(b) Give the marginal pdf of Y. (*Note*: As a way to check your work, I'll give you that $E(Y) = \mathit{Var}(Y) = 2$. But *do not submit any work related to using this check.*)

(c) For $y > 0$, give $E(X|Y = y)$. (*Note*: Your answer should be a rather simple function of y.)

(d) Give the value of $Cov(X, Y)$. (You can use the fact that $E(Y) = 2$ (since that value is given as a way to check your work for part (b)), but you may need to derive $E(X)$.)

(e) Give the cdf of $V = Y - X$.

(f) Give the variance of $V = Y - X$. (You can make use of your answer for part (d), along with the values of $Var(X)$ and $Var(Y)$. (You can use the fact that $Var(Y) = 2$ (since that value is given as a way to check your work for part (b)), but you may need to derive $Var(X)$.) Alternatively, you can make use of a correct answer for part (e) to obtain the desired variance. (My advice is to try to solve the problem using two different methods as a way to check your answer, but if you do this *only provide me with what you think is your best solution ... do not submit two solutions.*))