Solutions for Extra Ch. 7 Problems

1) Since there are n urns and n balls, the only way that none of the urns can be empty is for each urn to contain exactly one ball. Since the only ball that can possibly go into urn n is ball n, if none of the urns are to be empty, ball n must go into urn n. Since the only balls that can go into urn $n-1$ are balls $n-1$ and n, if none of the urns are to be empty, ball $n-1$ must go into urn $n-1$ (since it's already been established that if none of the urns are to be empty, ball n must go into urn n). Proceeding in a likewise manner, we can conclude that if none of the urns are to be empty, it must be that ball i must go into urn i, for $i = 1, 2, \ldots, n$. So, letting A_i denote the event that ball i goes into urn i, the desired probability is

$$P(\cap_{i=1}^n A_i) = \Pi_{i=1}^n P(A_i) = \Pi_{i=1}^n \frac{1}{i} = \frac{1}{n!}.$$

2) Letting A_i denote the event that urn i gets no balls, and $B_{i,j}$ denote the event that ball j misses urn i, noting that $P(B_{i,j}) = 1$ for $j < i$ and $P(B_{i,j}) = \frac{j-1}{j}$ for $j \geq i$, we have

$$P(A_i) = P(\cap_{j=1}^n B_{i,j}) = \Pi_{j=1}^n P(B_{i,j}) = \Pi_{j=i}^n \left(\frac{j-1}{j}\right) = \frac{i-1}{n}.$$

Letting X be the number of empty urns, and

$$I_i = \begin{cases} 1, & \text{if } A_i \text{ occurs,} \\ 0, & \text{otherwise,} \end{cases}$$

we have that the desired expectation is

$$E(X) = E\left(\sum_{i=1}^n I_i\right) = \sum_{i=1}^n P(A_i) = \sum_{i=1}^n \frac{i-1}{n} = \frac{1}{n}\sum_{i=1}^n (i-1) = \frac{1}{n}\sum_{i=0}^{n-1} i = \frac{1}{n}\frac{(n-1)n}{2} = \frac{n-1}{2}.$$

3) Letting A_j denote the event that the jth person gets a card matching his/her age, and noting that exactly one of the 1000 cards will have the number matching any of the people's age, we have $P(A_j) = 1/1000$, for $j = 1, 2, \ldots, 1000$. Letting X be the number of matches, and

$$I_j = \begin{cases} 1, & \text{if } A_j \text{ occurs,} \\ 0, & \text{otherwise,} \end{cases}$$

we have that the desired expectation is

$$E(X) = E\left(\sum_{j=1}^{1000} I_j\right) = \sum_{j=1}^{1000} P(A_j) = \sum_{j=1}^{1000} \frac{1}{1000} = 1.$$

4) It can be shown that

$$f_X(x) = (x + 1/2)\, I_{(0,\,1)}(x).$$

So

$$E(X) = \int_0^1 x\,(x+1/2)\,dx = \int_0^1 (x^2 + x/2)\,dx = (x^3/3 + x^2/4)|_0^1 = 1/3 + 1/4 = 7/12,$$

and by symmetry we also have $E(Y) = 7/12$. Since

$$\begin{aligned} E(XY) &= \int_0^1 \int_0^1 xy\,(x+y)\,dx\,dy \\ &= \int_0^1 \int_0^1 (x^2y + xy^2)\,dx\,dy \\ &= \int_0^1 (x^3y/3 + x^2y^2/2)|_0^1\,dy \\ &= \int_0^1 (y/3 + y^2/2)\,dy \\ &= (y^2/6 + y^3/6)|_0^1 \\ &= 1/3, \end{aligned}$$

it follows that the desired covariance is

$$E(XY) - E(X)E(Y) = 1/3 - (7/12)^2 = (48 - 49)/144 = -1/144.$$