Solutions for Additional Extra Problems From Ch. 7

1) For $x > 0$ we have

$$f_X(x) = \int_0^x 2e^{-2x}/x\,dy = 2e^{-2x},$$

and

$$f_{Y|X}(y|x) = [(2e^{-2x}/x)/(2e^{-2x})]I_{[0,\,x]}(y) = \frac{1}{x}I_{[0,\,x]}(y),$$

which is the pdf of a uniform $[0,\,x]$ random variable. It follows that the desired conditional expectation is just the mean of this uniform distribution, which is $x/2$.

2) Using Propostion 4.2 on p. 329 of the text, we have

$$\begin{aligned} Cov(Y_n, Y_{n+1}) &= Cov(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\ &= Cov(X_n, X_{n+1}) + Cov(X_n, X_{n+2}) + Cov(X_n, X_{n+3}) \\ &\quad + Cov(X_{n+1}, X_{n+1}) + Cov(X_{n+1}, X_{n+2}) + Cov(X_{n+1}, X_{n+3}) \\ &\quad + Cov(X_{n+2}, X_{n+1}) + Cov(X_{n+2}, X_{n+2}) + Cov(X_{n+2}, X_{n+3}) \\ &= 0 + 0 + 0 + \sigma^2 + 0 + 0 + 0 + \sigma^2 + 0 \\ &= 2\sigma^2. \end{aligned}$$

3) Letting Y be a random variable which equals 1 if a type 1 light bulb is chosen and equals 2 if a type 2 light bulb is chosen, we have

$$E(X) = E(E(X|Y)) = E(X|Y=1)p_Y(1) + E(X|Y=2)p_Y(2) = 1(1/2) + 2(1/2) = 3/2.$$

4) Letting X be the total number of flips required, and

$$Y = \begin{cases} 1, & \text{if 1st flip is Heads,} \\ 2, & \text{if 1st flip is Tails,} \end{cases}$$

we have

$$E(X) = E(E(X|Y)) = E(X|Y=1)p_Y(1) + E(X|Y=2)p_Y(2) = [1+1/(2/3)](1/3) + [1+1/(1/3)](2/3) = 7/2.$$

5) We have $P(M \le x|N=n) = [F(x)]^n = (1/2)^n$, and so

$$P(M \le x) = \sum_{n=1}^{\infty} P(M \le x|N=n)\,p_N(n) = \sum_{n=1}^{\infty} (1/2)^n\,(1/2)^n = \sum_{n=1}^{\infty} (1/4)^n = (1/4)/[1-(1/4)] = 1/3.$$

6) Letting

$$Y = \begin{cases} 1, & \text{if a type 1 bulb is selected,} \\ 2, & \text{if a type 2 bulb is selected,} \end{cases}$$

we have $p_Y(1) = p_Y(2) = 0.5$, so that $E(Y) = 1.5$, $E(Y^2) = 2.5$, and $\text{Var}(Y) = 0.25$. Since $E(X|Y) = \mu_Y = Y$ and $\text{Var}(X|Y) = \sigma_Y^2 = Y^2$, it follows (from the conditional variance formula) that

$$\text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y)) = E(Y^2) + \text{Var}(Y) = 2.5 + 0.25 = 2.75.$$

7) Using the mgfs it can be determined that $X \sim$ Poisson (2) and $Y \sim$ binomial (10, 3/4). (*Note*: The independence of X and Y is used for all three parts. It gives us that $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ and $E(XY) = E(X)E(Y)$.)

(a) The desired probability is

$$P(X+Y=2) = p_X(0)p_Y(2) + p_X(1)p_Y(1) + p_X(2)p_Y(0) \doteq 0.0000603.$$

(b) The desired probability is

$$\begin{aligned} P(XY=0) &= P(\{X=0\} \cup \{Y=0\}) \\ &= P(X=0) + P(Y=0) - P(\{X=0\} \cap \{Y=0\}) \\ &= e^{-2} + 4^{-10} - e^{-2}4^{-10} \\ &\doteq 0.135. \end{aligned}$$

(c) The desired expectation is

$$E(XY) = E(X)E(Y) = [2][10(3/4)] = 15.$$