Solutions for Extra Ch. 6 Problems

1) The joint pmf values can be found using the product rule for intersections:

$$p_{X_1,X_2}(x_1,x_2) = P(X_2 = x_2|X_1 = x_1)P(X_1 = x_1).$$

For example, we have

$$p_{X_1,X_2}(0,1) = P(X_2 = 1|X_1 = 0)P(X_1 = 0) = (4/11)(8/12) = 8/33.$$

Altogether, we have

$$p_{X_1,X_2}(x_1,x_2) = \begin{cases} 3/33, & (x_1,x_2) = (1,1), \\ 8/33, & (x_1,x_2) \in \{(1,0),(0,1)\}, \\ 14/33, & (x_1,x_2) = \{01,0), \\ 0, & \text{otherwise.} \end{cases}$$

2) Since Y must take a larger value than X does, for the distance $|Y - X|$ we have $Y - X$. The desired probability is

$$\begin{aligned} P(Y - X > L/4) &= P(Y > X + L/4) \\ &= 1 - P(Y < X + L/4) \\ &= 1 - \int_{L/4}^{L/2} \int_{L/2}^{x+L/4} (L/2)^{-2}\, dy\, dx \\ &= 7/8. \end{aligned}$$

3) In general,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx.$$

For $x \notin (0,\,1)$, $f_Y(y) = 0$. For $x \in (0,\,1)$,

$$f_Y(y) = \int_y^1 x^{-1}\, dx = \log x|_y^1 = -\log y.$$

Altogether, we have

$$f_Y(y) = -\log y\ I_{(0,\,1)}(y).$$

4) We have

$$\begin{aligned} P(X_1 < X_2) &= \int_0^{\infty} \int_{x_1}^{\infty} e^{-x_1} 2e^{-2x_2}\, dx_2\, dx_1 \\ &= \int_0^{\infty} e^{-x_1} \left(\int_{x_1}^{\infty} 2e^{-2x_2}\, dx_2 \right) dx_1 \\ &= \int_0^{\infty} e^{-x_1} \left(e^{-2x_1}\right) dx_1 \\ &= (1/3) \int_0^{\infty} 3e^{-3x_1}\, dx_1 \\ &= 1/3, \end{aligned}$$

where the last inegral equals 1 because it is the integral of a density of the support of the random variable.

5) Letting X_1 and X_2 be the profit in dollars for each of the next two weeks, we have that these are independent $\mathrm{N}\,(2200,\,(230)^2)$ random variables. So $X_1 + X_2$ is a $\mathrm{N}\,(4400,\,2(230)^2)$ random variable, and the desired probability is

$$\Phi((4000 - 4400)/\sqrt{2(230)^2}) = \Phi(-40/[23\sqrt{2}]) \doteq \Phi(-1.229751) \doteq 0.109.$$