Solutions for Extra Ch. 5 Problems

1) We have

$$\begin{aligned}
1 &= \int_0^\infty Cxe^{-x/2}\,dx \\
&= C\int_0^\infty xe^{-x/2}\,dx \\
&= C\left[-\frac{2x}{e^{x/2}}\Big|_0^\infty + \int_0^\infty 2e^{-x/2}\,dx\right] \\
&= C\left[0 + 4\int_0^\infty (1/2)e^{-x/2}\,dx\right] \\
&= C[0 + 4(1)] \\
&= 4C,
\end{aligned}$$

which gives us that $C = 1/4$. So for the desired probability we have

$$\begin{aligned}
\int_4^\infty (1/4)xe^{-x/2}\,dx &= (1/4)\int_4^\infty xe^{-x/2}\,dx \\
&= (1/4)\left[-\frac{2x}{e^{x/2}}\Big|_4^\infty + \int_4^\infty 2e^{-x/2}\,dx\right] \\
&= (1/4)\left[8/e^{4/2} + 4\int_4^\infty (1/2)e^{-x/2}\,dx\right] \\
&= (1/4)[8e^{-2} + 4(e^{-4/2})] \\
&= 3e^{-2}.
\end{aligned}$$

In both integrations above, integration by parts was used (with $u = 2x$, $v = -e^{-x/2}$, $du = 2\,dx$, and $dv = (1/2)e^{-x/2}\,dx$). L'Hôpital's rule was used to determine that $-x/e^{x/2}|_0^\infty = 0$. In the upper integration, the last integral equals 1 since it is the integral of an exponential random variable pdf over the support of the random variable.

2) Letting X be the number of points scored, the uniform (0, 10) distribution is used to obtain

$$p_X(10) = 0.1, \quad p_X(5) = 0.2, \quad p_X(3) = 0.2, \quad p_X(0) = 0.5.$$

So the desired expected value is

$$E(X) = 10(0.1) + 5(0.2) + 3(0.2) + 0(0.5) = 2.6.$$

3) Because of the "used is as good as new" property of exponential distibutions, the desired probability is just the probability that an exponential random variable having a mean of 8 assumes a value at least as large as 8, which is

$$\int_8^\infty (1/8)e^{-x/8}\,dx = -e^{-x/8}|_8^\infty = e^{-1} \doteq 0.368.$$

4) We have

$$
\begin{aligned}
E(|X-a|) &= \int_0^\infty |x-a|\lambda e^{-\lambda x}\,dx \\
&= \int_0^a (a-x)\lambda e^{-\lambda x}\,dx \;+\; \int_a^\infty (x-a)\lambda e^{-\lambda x}\,dx \\
&= a\int_0^a \lambda e^{-\lambda x}\,dx \;-\; \int_0^a x\lambda e^{-\lambda x}\,dx \;+\; \int_a^\infty x\lambda e^{-\lambda x}\,dx \;-\; a\int_a^\infty \lambda e^{-\lambda x}\,dx \\
&= a(1-e^{-\lambda a}) - \left[-\frac{x}{e^{\lambda x}}\Big|_0^a + \int_0^a e^{-\lambda x}\,dx\right] + \left[-\frac{x}{e^{\lambda x}}\Big|_a^\infty + \int_a^\infty e^{-\lambda x}\,dx\right] - ae^{-\lambda a} \\
&= a - ae^{-\lambda a} - \left[-\frac{a}{e^{\lambda a}} + (1/\lambda)\int_0^a \lambda e^{-\lambda x}\,dx\right] + \left[\frac{a}{e^{\lambda a}} + (1/\lambda)\int_a^\infty \lambda e^{-\lambda x}\,dx\right] - ae^{-\lambda a} \\
&= a - ae^{-\lambda a} - \left[-\frac{a}{e^{\lambda a}} + (1/\lambda)(1-e^{-\lambda a})\right] + \left[\frac{a}{e^{\lambda a}} + (1/\lambda)e^{-\lambda a}\right] - ae^{-\lambda a} \\
&= a - 1/\lambda + (2/\lambda)e^{-\lambda a}.
\end{aligned}
$$

Viewing this expression as a function of a and denoting it by $g(a)$, we have $g'(a)=0$ implies $a=\log 2/\lambda$. Since $g''(a)=2\lambda e^{-\lambda a}>0$, the solution of $g'(a)=0$ is a minimum. So the desired value of a is $\log 2/\lambda$.

5) We have $f_X(x)=e^{-x}\,I_{(0,\infty)}(x)$ and $Y=g(X)=\log X$. So $g(x)=\log x$ and $g^{-1}(x)=e^x$. Using Theorem 7.1 on p. 225 of the text, we have

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| \\
&= f_X(e^y)\left|\frac{d}{dy}e^y\right| \\
&= e^{-e^y}\left|e^y\right| \\
&= e^{y-e^y}.
\end{aligned}
$$

6) We have

$$
1 = \int_0^\infty ax^2e^{-bx^2}\,dx.
$$

Letting $y=bx^2$ (from which it follows that $dx = dy/(2\sqrt{by})$, we have

$$
\begin{aligned}
1 &= \int_0^\infty a\frac{y}{b}e^{-y}\frac{1}{2\sqrt{by}}\,dy \\
&= \frac{a}{2b^{3/2}}\int_0^\infty y^{1/2}e^{-y}\,dy \\
&= \frac{a}{2b^{3/2}}\Gamma(3/2) \\
&= \frac{a}{2b^{3/2}}(1/2)\Gamma(1/2) \\
&= \frac{a\sqrt{\pi}}{4b^{3/2}},
\end{aligned}
$$

from which it follows that

$$
a = \frac{4b^{3/2}}{\sqrt{\pi}}.
$$