Soultions for Extra Ch. 4 and Ch. 5 Problems

1) Letting X be the number of suicides in a month, X is a Poisson random variable having parameter/mean $(1/100,000)(400,000) = 4$. The desired probability is

$$
\begin{aligned}
P(X \geq 8) &= 1 - P(X \leq 7) \\
&= 1 - \sum_{x=0}^{7} \frac{4^x}{x!} e^{-4} \\
&= 1 - (1 + 4 + 4^2/2 + 4^3/6 + 4^4/24 + 4^5/120 + 4^6/720 + 4^7/5040)e^{-4} \\
&= 1 - (16319/315)e^{-4} \\
&\doteq 0.0511.
\end{aligned}
$$

2) The possible values of X are $0, 1, 2, \ldots$. For $x = 0, 1, 2, \ldots$, we can express the event that $X = x$ as the intersection of A_x and B_x, where A_x is the event that 9 Heads and x Tails occur in the first $x + 9$ trials, and B_x is the event that a Head occurs on the $(x + 10)$th trial. Since A_x and B_x should be assumed to be independent, we have

$$P(X = x) = \left[\binom{x+9}{9} (1/2)^9 (1/2)^x \right] [1/2],$$

and so the desired pmf can be expressed as

$$p_X(x) = \left[\binom{x+9}{x} / 2^{x+10} \right] I_{\{0,1,2,\ldots\}}(x).$$

3) X is a hypergeometric random variable. We have

$$P(X = 0) = \frac{\binom{6}{0}\binom{94}{10}}{\binom{100}{10}} = \frac{1886711}{3612280} \doteq 0.522.$$

4) Letting Y be the number of boxes getting exactly one ball, we can express Y as the sum $I_1 + I_2 + \cdots + I_5$, where I_j is a Bernoulli (p) random variable ($j = 1, 2, \ldots, 5$), where p is the probability of a box getting exactly one ball. Making use of the binomial (10, 1/5) distribution, we have

$$p = \binom{10}{1} (1/5)(4/5)^9 = 10(4^9)/5^{10}.$$

Since the expected value of each of the Bernoulli random variables is just p, it follows that the desired expected value is

$$E(Y) = E(I_1) + \cdots + E(I_5) = 5p = 10(4/5)^9 \doteq 1.34.$$

5) We have

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)\, dx \\
&= \int_{-1}^{1} c(1 - x^2)\, dx \\
&= c(x - x^3/3)|_{-1}^{1} \\
&= c(4/3),
\end{aligned}
$$

which implies that $c = 3/4$. Letting F denote the cdf, since the support of X is $(-1,\,1)$, we have $F(x) = 0$ for $x \le -1$, and $F(x) = 1$ for $x \ge 1$. For $x \in (-1,\,1)$,

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_{-1}^{x} (3/4)(1-t^2)\,dt \\ &= (3/4)(t - t^3/3)|_{-1}^{x} \\ &= (3/4)[(x - x^3/3) - (-1 + 1/3)] \\ &= \frac{3}{4}x - \frac{x^3}{4} + \frac{1}{2}. \end{aligned}$$

Altogether, we have

$$F(x) = \begin{cases} 1, & x \ge 1, \\ \frac{3}{4}x - \frac{x^3}{4} + \frac{1}{2}, & -1 < x < 1, \\ 0, & x \le -1. \end{cases}$$

6) We have

$$P(X > 25) = \int_{25}^{\infty} f(x)\,dx = \int_{25}^{\infty} 10x^{-2}\,dx = 2/5.$$

7) Letting the answer to part (a) be denoted by p, the number of months in which there will be at least 8 suicides can be assumed to be a binomial $(12,\,p)$ random variable, which I'll call Y. The desired probability is

$$\begin{aligned} P(Y \ge 2) &= 1 - P(Y \le 1) \\ &= 1 - \binom{12}{0} p^0 (1-p)^{12} + \binom{12}{1} p^1 (1-p)^{11} \\ &= 1 - (1-p)^{12} - 12p(1-p)^{11} \\ &\doteq 0.123, \end{aligned}$$

where $p = 1 - (16319/315)e^{-4}$ (which was obtained above in the solution for Problem 1).

8) X is a hypergeometric random variable. We have

$$P(X > 2) = 1 - \sum_{x=0}^{2} \frac{\binom{6}{x}\binom{94}{10-x}}{\binom{100}{10}} \doteq 0.0126.$$

9) Letting Y be the number of boxes getting no balls, we can express Y as the sum $I_1 + I_2 + \cdots + I_5$, where I_j is a Bernoulli (p) random variable $(j = 1, 2, \ldots, 5)$, where p is the probability of a box getting no balls. Making use of the binomial $(10,\,1/5)$ distribution, we have

$$p = \binom{10}{0} (1/5)^0 (4/5)^{10} = (4/5)^{10}.$$

Since the expected value of each of the Bernoulli random variables is just p, it follows that the desired expected value is

$$E(Y) = E(I_1) + \cdots + E(I_5) = 5p = 5(4/5)^{10} = 4^{10}/5^9 \doteq 0.537.$$

10) We have for $x \le 10$ we have $F_X(x) = P(X \le x) = 0$. For $x > 10$ we have

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{10}^{x} 10t^{-2}\,dt = 1 - 10/x.$$

So altogether we have

$$F_X(x) = \begin{cases} 1 - 10/x, & x > 10, \\ 0, & x \le 10. \end{cases}$$