⟨ Problem 11.1 & Problem 11.7 of E&T ⟩

If $\hat{\theta} = \bar{x}$, we have

$$\hat{\theta}_{(i)} = \frac{(\sum_{j=1}^{n} x_j) - x_i}{n-1} = \frac{n}{n-1}\bar{x} - \frac{1}{n-1}x_i.$$

It follows that

$$\hat{\theta}_{(\cdot)} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{(i)} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{n}{n-1}\bar{x} - \frac{1}{n-1}x_i\right) = \frac{n}{n-1}\bar{x} - \frac{1}{n-1}\bar{x}$$

So we have

$$\hat{\theta}_{(i)} - \hat{\theta}_{(\cdot)} = \left(\frac{n}{n-1}\bar{x} - \frac{1}{n-1}x_i\right) - \left(\frac{n}{n-1}\bar{x} - \frac{1}{n-1}\bar{x}\right) = \frac{1}{n-1}(\bar{x} - x_i),$$

and

$$\begin{aligned}\widehat{se}_{jack} &= \left\{\frac{n-1}{n}\sum_{i=1}^{n}[\hat{\theta}_{(i)} - \hat{\theta}_{(\cdot)}]^2\right\}^{1/2} \\ &= \left\{\frac{n-1}{n}\sum_{i=1}^{n}\left[\frac{1}{n-1}(\bar{x} - x_i)\right]^2\right\}^{1/2} \\ &= \left\{\frac{n-1}{n}\sum_{i=1}^{n}\left[\frac{1}{(n-1)^2}(x_i - \bar{x})^2\right]\right\}^{1/2} \\ &= \left\{\frac{1}{(n-1)n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right\}^{1/2},\end{aligned}$$

which equals (11.8), and thus Problem 11.1 has been solved.

Above we have $\hat{\theta}_{(\cdot)} = \frac{n}{n-1}\bar{x} - \frac{1}{n-1}\bar{x}$, which equals $\bar{x}$. Thus $\hat{\theta}_{(\cdot)} - \hat{\theta} = \bar{x} - \bar{x} = 0$, and thus Problem 11.7 has been solved.

⟨ Problem 11.8 & Problem 10.8 of E&T ⟩

If $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$, we have (treating the x_i as iid r.v.'s and noting that $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is unbiased for σ^2)

$$\text{bias}(\hat{\theta}) = E(\hat{\theta}) - \sigma^2 = E\left(\frac{n-1}{n}S^2\right) - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{\sigma^2}{n},$$

and thus part (a) of Problem 11.8 has been solved.

We have, letting $\overline{x}_{(i)}$ denote $\frac{1}{n-1}\sum_{j\neq i} x_j$, and $\hat{m}_2 = \frac{\sum_{j=1}^n x_j^2}{n}$,

$$\begin{aligned}
\hat{\theta}_{(i)} &= \frac{1}{n-1}\sum_{j\neq i}(x_j - \overline{x}_{(i)})^2 \\
&= \frac{1}{n-1}\sum_{j\neq i} x_j^2 - \overline{x}_{(i)}^2 \\
&= \frac{1}{n-1}\left[\left(\sum_{j=1}^n x_j^2\right) - x_i^2\right] - \left\{\frac{1}{n-1}\left[\left(\sum_{j=1}^n x_j\right) - x_i\right]\right\}^2 \\
&= \frac{n}{n-1}\hat{m}_2 - \frac{1}{n-1}x_i^2 - \left\{\frac{n}{n-1}\overline{x} - \frac{1}{n-1}x_i\right\}^2 \\
&= \frac{n}{n-1}\hat{m}_2 - \frac{1}{n-1}x_i^2 - \left\{\frac{n^2}{(n-1)^2}\overline{x}^2 - \frac{2n}{(n-1)^2}\overline{x}x_i + \frac{1}{(n-1)^2}x_i^2\right\} \\
&= \frac{n}{n-1}\hat{m}_2 - \left[\frac{1}{n-1} + \frac{1}{(n-1)^2}\right]x_i^2 - \frac{n^2}{(n-1)^2}\overline{x}^2 + \frac{2n}{(n-1)^2}\overline{x}x_i \\
&= \frac{n}{n-1}\hat{m}_2 - \frac{n}{(n-1)^2}x_i^2 - \frac{n^2}{(n-1)^2}\overline{x}^2 + \frac{2n}{(n-1)^2}\overline{x}x_i.
\end{aligned}$$

It follows that

$$\begin{aligned}
\hat{\theta}_{(\cdot)} &= \frac{n}{n-1}\hat{m}_2 - \frac{n}{(n-1)^2}\hat{m}_2 - \frac{n^2}{(n-1)^2}\overline{x}^2 + \frac{2n}{(n-1)^2}\overline{x}^2 \\
&= \left[\frac{n}{n-1} - \frac{n}{(n-1)^2}\right]\hat{m}_2 - \left[\frac{n^2}{(n-1)^2} - \frac{2n}{(n-1)^2}\right]\overline{x}^2.
\end{aligned}$$

Since $\hat{\theta} = \hat{m}_2 - \overline{x}^2$, we have

$$\begin{aligned}
\widehat{bias}_{jack} &= (n-1)\left\{\hat{\theta}_{(\cdot)} - \hat{\theta}\right\} \\
&= (n-1)\left\{\left[\frac{n}{n-1} - \frac{n}{(n-1)^2}\right]\hat{m}_2 - \left[\frac{n^2}{(n-1)^2} - \frac{2n}{(n-1)^2}\right]\overline{x}^2 - \hat{m}_2 + \overline{x}^2\right\} \\
&= (n-1)\left\{\left[\frac{n}{n-1} - \frac{n}{(n-1)^2} - 1\right]\hat{m}_2 - \left[\frac{n^2}{(n-1)^2} - \frac{2n}{(n-1)^2} - 1\right]\overline{x}^2\right\} \\
&= (n-1)\left\{-\frac{1}{(n-1)^2}\hat{m}_2 - \frac{1}{(n-1)^2}\overline{x}^2\right\} \\
&= -\frac{1}{(n-1)}\left\{\hat{m}_2 - \overline{x}^2\right\} \\
&= -\frac{1}{(n-1)}\hat{\theta} \\
&= -\frac{1}{(n-1)}\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2 \\
&= -s^2/n,
\end{aligned}$$

and thus part (b) of Problem 11.8 has been solved.

If $\hat{\theta}$ is bias corrected, we have

$$\begin{aligned}
\overline{\theta} &= \hat{\theta} - \widehat{bias}_{jack} \\
&= \frac{\sum_{i=1}^n (x_i - \overline{x})^2}{n} - \left(-\frac{s^2}{n}\right) \\
&= \frac{n-1}{n}s^2 + \frac{s^2}{n} \\
&= s^2,
\end{aligned}$$

and thus Problem 10.8 has been solved.