Solutions for
Midterm Exam 1

Instructions: You cannot use any books or notes. You can use a calculator, but not a computer. You are not allowed to communicate with anyone (verbally, in writing, or electronically), except for me, during the exam period (and at most I will clarify a problem — I won't give you any help with solving the problems).

The exam consists of 5 problems, having a total of 6 parts. Each of the 6 parts is worth 20 points, and I'll count your best 5 of 6 scores from these parts to determine your overall exam score.

I used this exam over 5 years ago.

Express probabilities as exact values (as fractions, or in decimal form), or else round them to three significant digits. (Note: 0.00402 has three significant digits, and 0.004 only has one significant digit.) In order to receive full credit, do not express final answers as expressions that need to be evaluated. (So a final answer should not include a binomial coefficient or even a factorial.)

The red refers to the class over 5 years ago.

Be sure to *use notation and terminology properly!* For most of the parts which follow, I want you to *define events* (if they aren't already defined in the statement of the problem) and *provide justification* for your answers. (If a part can be solved just using Ch. 1 and Ch. 2 results, you don't necessarily have to define events, but when using Ch. 3 results you should define events.) Put all of your work on these sheets. If you need more room, direct me to look for additional work on the back of a page. **Draw boxes around your final answers!**

7 of 22 got right answer

1) Suppose that an ordinary, fair, 6-sided die will be rolled three times. Making the usual assumptions about rolling a die multiple times, what is the probability that all three rolls will result in 6 spots on the upward face of the die, given that at least one of the three rolls will result in 6 spots on the upward face of the die?

15 of 49 got it in 2022

Letting A_i denote the event that roll i results in a ⚅, the desired probability is

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3 \mid A_1 \cup A_2 \cup A_3) &= \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cup A_2 \cup A_3)} \\
&= \frac{P(A_1 \cap A_2 \cap A_3)}{1 - P((A_1 \cup A_2 \cup A_3)^c)} \\
&\overset{\text{DeMorgan}}{=} \frac{P(A_1 \cap A_2 \cap A_3)}{1 - P(A_1^c \cap A_2^c \cap A_3^c)} \\
&\overset{\text{ind.}}{=} \frac{P(A_1)P(A_2)P(A_3)}{1 - P(A_1^c)P(A_2^c)P(A_3^c)} \\
&= \frac{(\frac{1}{6})^3}{1 - (\frac{5}{6})^3} \\
&= \frac{1}{216 - 125} \\
&= \boxed{\frac{1}{91}}.
\end{aligned}
$$

(Alternatively, from a sample space of $6^3 = 216$ equally-likely outcomes, one can ignore the $5^3 = 125$ outcomes containing only the results ⚀ through ⚄, and focus on a reduced sample space consisting of the $216 - 125 = 91$ outcomes containing at least one ⚅. Of these, only one of them contains all ⚅s, and so the desired cond'l prob. is $1/91$.)

If you didn't use definition of conditional probability or a reduced sample space point of view I didn't give you much credit for an incorrect answer. (Some tried to take a "shortcut" similar to the one referred to on p. 3-5 of the class notes. But such a short cut isn't a legitimate method.)

2) Consider a bag of four tennis balls, three of which are brand new, and one of which has been used. Suppose that a ball is randomly selected from amongst the four balls. Then the ball is used and returned to the bag. (At this point the bag will either contain one or two used balls. If the randomly selected ball was new, it has now been used, and so in this case when it is returned the bag will contain two used balls and two new ones. On the other hand, if the randomly selected ball was the one used one, upon returning it to the bag the bag will contain only one used ball along with three new ones.) Finally, once the ball originally selected is returned to the bag, a second random selection will be made from the four balls in the bag.

13 of 22 got it right

31 of 49 got it in 2022

(a) What is the probability that *both* random selections will result in a new ball? For this part, I definitely want you to define some events and make use of a simple result from Ch. 3 of the text.

Some ignored this.

Letting N_i denote the event that draw i results in a new ball, the desired probability is

$$P(N_1 \cap N_2) = P(N_2 \mid N_1) P(N_1) = \left(\tfrac{2}{4}\right)\left(\tfrac{3}{4}\right) = \boxed{\tfrac{3}{8}}.$$

none of these steps should have been omitted

Notice how this short simple solution is written as a single sentence. (Some students have a tendency to have many scattered pieces instead of a straightforward "linear" solution.)

14 of 22 got it right

25 of 49 got it in 2022

(b) What is the probability that *the second* random selection will result in a new ball? (*Hint*: Noting that this would be trivial to answer if you knew whether the first ball randomly selected was new or used, does this fact suggest the use of a particular result from Ch. 3 of the text?)

Using the events defined for part (a), the desired probability is

$$P(N_2) = P(N_2 \mid N_1) P(N_1) + P(N_2 \mid N_1^c) P(N_1^c) = \left(\tfrac{2}{4}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) = \boxed{\tfrac{9}{16}}.$$

- Notice again a simple one sentence "linear" solution.
- On the exam review web page I had in bold font that you should look at an example in the notes almost identical to this part. No one should have missed this part!

10 of 22 got it correct

8 of 49 got it in 2022

3) Suppose a dash is transmitted with probability 0.6 and a dot is transmitted with probability 0.4. If a dash or dot is received correctly with probability 0.9 and received incorrectly (meaning a dash is received as a dot, or a dot is received as a dash) with probability 0.1, what is the probability that a dash is sent given that a dash is received?

Letting

D_s be the event that a dash is sent &

D_r be the event that a dash is received,

for the desired probability we have

these are the only events needed — some students defined other events that led to solutions that were incorrect and/or hard to follow

$$P(D_s \mid D_r) \overset{\text{Bayes}}{=} \frac{P(D_r \mid D_s)P(D_s)}{P(D_r \mid D_s)P(D_s) + P(D_r \mid D_s^c)P(D_s^c)}$$

$$= \frac{(0.9)(0.6)}{(0.9)(0.6) + (0.1)(0.4)}$$

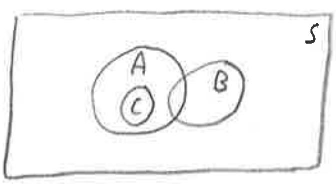
$$= \boxed{\frac{27}{29}}.$$

I strongly hinted in the Announcements section of the Blackboard site that Bayes formula should be used for one of the parts of your exam.

10 of 22 got it correct

17 of 49 got it in 2022

4) Suppose that A, B, and C are events such that $P(A) = 0.3$, $P(B) = 0.2$, $P(C) = 0.1$, A and B are independent, B and C are mutually exclusive, and $C \subset A$. Give the value of $P(A^C \cap B^C \cap C^C)$. (*Hint*: Use one of DeMorgan's laws, along with the inclusion-exclusion result.)



$$P(A^c \cap B^c \cap C^c) \overset{\text{DeMorgan}}{=} P((A \cup B \cup C)^c)$$

$$= 1 - P(A \cup B \cup C)$$

$$= 1 - [P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)]$$

$$\overset{\text{ind.}}{=} 1 - [P(A) + P(B) + P(C) - P(A)P(B) - P(C) - P(\emptyset) + P(\emptyset)]$$

$$= 1 - [P(A) + P(B) - P(A)P(B)]$$

$$= 1 - [0.3 + 0.2 - (0.3)(0.2)]$$

$$= \boxed{0.56}.$$

15 of 22 got it Correct

28 of 49 got it in 2022

5) Suppose that 3 men and 5 women will be randomly put into a linear arrangement, from left to right, with all 8! possibilities being equally likely? What is the probability that all of the women will be next to one another and all of the men will be next to one another?

There are $(3!)(5!)$ permutations with the men in the first 3 positions and the women in the last 5 positions, and another $(3!)(5!)$ permutations with the women in the first 5 positions and the men in the last 3 positions. Altogether, the desired probability is

$$\frac{2(3!)(5!)}{8!} = \frac{2}{\binom{8}{3}} = \boxed{\frac{1}{28}}.$$

An alternative explanation for this is that there are $\binom{8}{3}$ sets of 3 locations for the men, and only 2 of them, $\{1,2,3\}$ & $\{6,7,8\}$, correspond to all of the men and all of the women being next to one another.