APPENDIX A: PROOFS FOR "MANAGING PRODUCT ROLLOVERS" BY KOCA, SOUZA AND DRUEHL

Derivation of $h(\gamma)$

Let R_{it} denote the reservation price for product i at time t, a random variable. We write $R_{1t} = u(\Omega)\varepsilon_{1t}$, where $u(\cdot)$ is a (deterministic) *linear* mapping function and ε_{1t} is a random variable with a Weibull distribution (so that R_{it} has a Weibull distribution); similarly $R_{2t} = u((1+\gamma)\Omega)\varepsilon_{2t}$. We define customer utility $\mathcal{V}_{it}$ as a log function of the customer's reservation price $\mathcal{V}_{it} = \ln(R_{it}) = \ln(u(\Omega)) + \ln(\varepsilon_{it})$. Because ε_{it} has a Weibull distribution, $\mathcal{V}_{it}$ has a Gumbel distribution; this is consistent with the Logit model for choice. As a result, the probability that a customer adopts the new generation is

$$h(\gamma) = \frac{e^{\ln(u((1+\gamma)\Omega))}}{e^{\ln(u((1+\gamma)\Omega))} + e^{\ln(u(\Omega))}} = \frac{u((1+\gamma)\Omega)}{u(\Omega) + u((1+\gamma)\Omega)} = \frac{1+\gamma}{2+\gamma}. \tag{A.1}$$

Proof of Proposition 1

For a non-stationary Poisson process with intensity $\Lambda(t)$, $\mathrm{E}\left[N\left(\Lambda(t)\right)\right] = \Lambda(t)$, and thus (6) becomes

$$\begin{aligned} \max_{p_2(t)} \Pi_2^j &= \int_{\tau}^{(1+\alpha)\tau} e^{-\delta(t-\alpha\tau)} \left(p_2(t) - c_p\right) \mathrm{E}\left[N\left(\lambda_2^j(t)\bar{G}_{2t}(p_2)\right)\right] dt \\ &= \int_{\tau}^{(1+\alpha)\tau} e^{-\delta(t-\alpha\tau)} \left(p_2(t) - c_p\right) \lambda_2^j(t)\bar{G}_{2t}(p_2) dt \end{aligned} \tag{A.2}$$

This is a simple optimal control problem, with the first-order necessary condition given in (7). The solution is similar to that found in Bitran and Mondschein (1997). For uniqueness of the solution to (7), we need

$$K_{2,t} = p_2(t) - \frac{\bar{G}_{2t}}{G_{2t}}$$

to be an increasing function of $p_2(t)$ since $\lim_{p_2(t)\to\infty} K_{2,t} = \infty$. Therefore, we need

$$0 < \frac{dK_{2,t}}{dp_2} = 1 - \frac{d}{dp_2}\frac{\bar{G}_{2t}}{G_{2t}} = 1 - \frac{-G_{2t}^2 - \bar{G}_{2t}g'_{2,t}}{G_{2t}^2}$$

which becomes

$$0 > \frac{-2G_{2t}^2 \bar{G}_{2t} - \bar{G}_{2t}^2 g'_{2,t}}{G_{2t}^2} = \frac{d}{dp_2} \frac{\bar{G}_{2t}^2}{G_{2t}} \quad \blacksquare$$

Proof of Proposition 2

We show through fluid approximations (Mandelbaum & Pats, 1998) that the solutions to the deterministic version of (5) is asymptotically optimal as initial maximum arrival rate, M_0, and I_0 grow proportionally large. However, since I_0 is a decision variable, we first show that it is optimal to select I_0 proportionally large as M_0.

Consider a sequence of instances of problem (5) indexed by $n \in \mathbb{Z}_+$. Let M_0^n denote the initial maximum arrival rate and λ_1^{jn} be the resulting arrival rate intensity function for the n^{th} instance. Let

$$\lim_{n\to\infty} \frac{M_0^n}{n} = M_0.$$

Thus, we have

$$\lim_{n\to\infty} \frac{\lambda_1^{jn}}{n} = \lambda_1^j.$$

Let I_0^n be the decision parameter for the final build and $I^n(t)$ denote the corresponding inventory trajectory for the n^{th} instance, and let all other parameters be held constant, independent of n. For the n^{th} instance, (5) becomes

$$\begin{aligned} \max_{I_0^n, p_1(t)} \mathrm{E}\Bigg[&- \int_{\alpha\tau}^{T^j} e^{-\delta(t-\alpha\tau)} p_1(t) dI^n(t) + e^{-\delta(1-\alpha)\tau} c_s \left(I_0^n + \int_{\alpha\tau}^{T^j} dI^n(t) \right) \\ &- c_h \int_{\alpha\tau}^{T^j} e^{-\delta(t-\alpha\tau)} I^n(t) dt - c_p I_0^n \Bigg] \end{aligned} \tag{A.3}$$

s.t.

$$- \int_{\alpha\tau}^{T^j} dI^n(t) \leqslant I_0^n$$

$$I^n(t) = I_0^n - N \left(\int_{\alpha\tau}^{t} \lambda_1^{jn}(u) \bar{G}_{1u}(p_1) du \right) \text{ for } t \in \left[\alpha\tau, T^j\right],$$

where we wrote $I(T^j) \geqslant 0$ as $- \int_{\alpha\tau}^{T^j} dI(t) \leqslant I_0$. After dividing the second constraint by n, taking limits on both sides, and applying Lebesgue's monotone convergence theorem, we get

$$\lim_{n\to\infty}\frac{1}{n}I^n(t)=\lim_{n\to\infty}\frac{1}{n}I_0^n-N\left(\int\limits_{\alpha\tau}^{t}\lambda_1^j(u)\bar{G}_{1u}(p_1)du\right).$$

Similarly, from the first constraint in (A.3), we have

$$\lim_{n\to\infty}-\frac{1}{n}\int_{\alpha\tau}^{T^j}dI^n(t)\leqslant\lim_{n\to\infty}\frac{1}{n}I_0^n.$$

Therefore, applying the same transformation to the objective function, we can rewrite (A.3) as

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\max_{I_0^n,p_1(t)}\mathrm{E}\Bigg[&-\int_{\alpha\tau}^{T^j}e^{-\delta(t-\alpha\tau)}p_1(t)dI^n(t)+e^{-\delta(1-\alpha)\tau}c_s\left(I_0^n+\int_{\alpha\tau}^{T^j}dI^n(t)\right)\\&-c_h\int_{\alpha\tau}^{T^j}e^{-\delta(t-\alpha\tau)}I^n(t)dt-c_pI_0^n\Bigg]\\ \text{s.t.}\qquad&\\ \lim_{n\to\infty}-\frac{1}{n}\int_{\alpha\tau}^{T^j}dI^n(t)&\leqslant\lim_{n\to\infty}\frac{1}{n}I_0^n\\ \lim_{n\to\infty}\frac{1}{n}I^n(t)&=\lim_{n\to\infty}\frac{1}{n}I_0^n-N\left(\int_{\alpha\tau}^{t}\lambda_1^j(u)\bar{G}_{1u}(p_1)du\right)\ \text{ for } t\in\left[\alpha\tau,T^j\right].\end{aligned}\tag{A.4}$$

Suppose (I_0^*,p_1^*) is an optimal solution to (5), with the optimal objective function value π_1^*. Then, (I_0^{n*},p_1^*) is an optimal solution to (A.4) with the objective function value π_1^{n*}, such that I_0^{n*} and π_1^{n*} satisfy $\lim_{n\to\infty}I_0^{n*}/n=I_0^*$ and $\pi_1^{n*}=\pi_1^*$, respectively. This follows by observing that (A.4) is equivalent to problem (5) divided by n and taking limits as $n\to\infty$. As a result, we have shown that it is optimal to let the final build, I_0, grow proportionally large as M_0 in the asymptotic regime.

Noting that the demand intensity process

$$\int_{\alpha\tau}^{t}\lambda_1^j(u)\bar{G}_{1u}(p_1)du$$

is continuous and uniformly bounded in $[\alpha\tau,T^j]$, and we find that in the limit as $n\to\infty$, $I^n(t)/n$ converges (almost surely and uniformly over a compact set) to $I(t)$, given by

$$I(t)=I_0-\int_{\alpha\tau}^{t}\lambda_1^j(u)\bar{G}_{1u}(p_1)du.$$

Further details regarding the proof of this convergence result can be found in Mandelbaum and Pats (1998). In this asymptotic regime, the stochastic optimization problem in (5) reduces to the optimal control problem in (8), where $I_0 + \int_{\alpha\tau}^{T^j} dI(t)$ is replaced with $I(T^j)$, and the second constraint is substituted into the first term in the objective function.

The solution to (8) can be found as follows. Treating $I(t)$ as the state variable and $p_1(t)$ as the control variable, and letting ν and $\omega(t)$ be the multipliers for the first and second constraints in (8), the Hamiltonian is $H = e^{-\delta(t-\alpha\tau)}(\lambda_1^j \bar{G}_{1t} p_1 - c_h I) - \omega \lambda_1^j \bar{G}_{1t}$, where arguments have been suppressed for simplicity. The optimality conditions are:

$$\frac{\partial H}{\partial p_1} = 0 \ \therefore\ \lambda_1^j \left[e^{-\delta(t-\alpha\tau)} \left(-p_1 G_{1t} + \bar{G}_{1t} \right) + \omega(t) G_{1t} \right] = 0, \tag{A.5}$$

$$\frac{\partial H}{\partial I} = -\frac{\partial \omega}{\partial t} \ \therefore\ c_h e^{-\delta(t-\alpha\tau)} = \frac{\partial \omega}{\partial t}, \tag{A.6}$$

$$\omega(T^j) = \nu + e^{-\delta\left(T^j - \alpha\tau\right)} c_s \text{ and } \nu I(T^j) = 0. \tag{A.7}$$

A first-order condition for I_0 is obtained by considering that the marginal revenue from the last unit must equal to its marginal cost (including the procurement cost and cumulative holding costs in time). That is,

$$\omega(T^j) = c_p + c_h \int_{\alpha\tau}^{T^j} e^{-\delta(u-\alpha\tau)} du. \tag{A.8}$$

Combining (A.7) and (A.8), we get

$$\nu = c_p + c_h \int_{\alpha\tau}^{T^j} e^{-\delta(u-\alpha\tau)} du - e^{-\delta\left(T^j - \alpha\tau\right)} c_s. \tag{A.9}$$

However, we must have $\nu > 0$, otherwise $I_0 \to \infty$ is optimal and the problem in (8) is unbounded. Therefore, from (A.7), $I(T^j) = 0$. In other words, the entire initial inventory is depleted during the sales horizon. To find $p_1(t)$, we proceed as follows. From (A.5),

$$\omega(t) = e^{-\delta(t-\alpha\tau)} \left(p_1 - \frac{\bar{G}_{1t}}{G_{1t}} \right). \tag{A.10}$$

On the other hand, (A.7) and (A.8) yield

$$\omega(t) = c_p + c_h \int_{\alpha\tau}^{t} e^{-\delta(u-\alpha\tau)} du. \tag{A.11}$$

We combine (A.10) and (A.11) to obtain the necessary condition for the optimal price pattern for product 1, given in (9). The proof of uniqueness follows the same steps as in the proof for Proposition 1.

Once the optimal price path is determined using (9), and given that $I(T^j) = 0$, the optimal initial inventory is equal to the total sales through the planning horizon. ∎

The Normalization of ϕ for the Regression

We normalize the parameter ϕ, for the purposes of running the regression, so that it takes values between 0 and 1, instead of between 0 and ∞. We do this by mapping ϕ to a new parameter θ, according to the normalizing relationship:

$$\phi = (1-\alpha)\tau \Big/ \left(\frac{1}{\theta} + \mathrm{W}\left(-\frac{1}{\theta} \times e^{-\frac{1}{\theta}} \right) \right), \tag{A.12}$$

where $\mathrm{W}(\cdot)$ is the Lambert W function. The Lambert W function is the inverse of $f(w) = we^w$ and we use the zeroth branch which is single valued and real for the range of θ considered. It is easily verified that $\lim_{\phi\to\infty} \theta = 1$, and $\lim_{\phi\to 0} \theta = 0$.

References

Bitran, G. R., & Mondschein, S. V. (1997). Periodic pricing of seasonal products in retailing. *Management Science*, *43*(1), 64–79.

Mandelbaum, A., & Pats, G. (1998). State-dependent stochastic networks. Part I: Approximations and applications with continuous diffusion limits. *Annals of Applied Probability*, *8*(2), 569–646.

Table A.1: Statistics of multiple linear regression: two-way interaction effects.

Factor	t-value	
(intercept)	-2.7	$**$
τ	-60.9	$***$
α	9.0	$***$
$\log(k)$	-3.8	$***$
μ	35.3	$***$
γ	-35.7	$***$
$p+q$	-43.7	$***$
$\log(q/p)$	68.3	$***$
ϕ	-94.5	$***$
$\tau : \alpha$	-20.9	$***$
$\tau : \log(k)$	-17.7	$***$
$\tau : \mu$	-1.8	$\circ$
$\tau : \gamma$	-35.0	$***$
$\tau : p+q$	-30.4	$***$
$\tau : \log(q/p)$	25.1	$***$
$\tau : \phi$	-48.1	$***$
$\alpha : \log(k)$	1.9	$\circ$
$\alpha : \mu$	6.3	$***$
$\alpha : \gamma$	-12.0	$***$
$\alpha : p+q$	-20.0	$***$
$\alpha : \log(q/p)$	30.7	$***$
$\alpha : \phi$	-9.5	$***$
$\log(k) : \mu$	10.0	$***$
$\log(k) : \gamma$	-10.3	$***$
$\log(k) : p+q$	-11.3	$***$
$\log(k) : \log(q/p)$	17.7	$***$
$\log(k) : \phi$	-24.4	$***$
$\mu : \gamma$	-4.0	$***$
$\mu : p+q$	-11.2	$***$
$\mu : \log(q/p)$	15.7	$***$
$\mu : \phi$	-21.9	$***$
$\gamma : p+q$	-16.3	$***$
$\gamma : \log(q/p)$	25.8	$***$
$\gamma : \phi$	-35.2	$***$
$p+q : \log(q/p)$	-6.3	$***$
$p+q : \phi$	-4.2	$***$
$\log(q/p) : \phi$	-6.0	$***$
Adj. R-sq.	0.837	

Significance codes: '$***$': $p \approx 0$; '$**$': $0.001 < p < 0.01$; '$\circ$': $0.05 < p < 0.1$