Projection for Creating an OctaGlobe

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1 Dimensions of OctaGlobe

The OctaGlobe is a polyhedron composed of faces which are rectangles, trapezoids, and octagons (Fig. 1a). Each face represents a sector of a sphere bounded by given circles of longitude and latitude. Octagons represent longitude circles (Fig. 1b) and latitude circles (fig. 1c). Therefore each face represents 1/8th of a circle or 45^o in latitude and longitude, except for the polar faces, which contain all longitudes within 22.5^o of the poles (Fig. 1c).

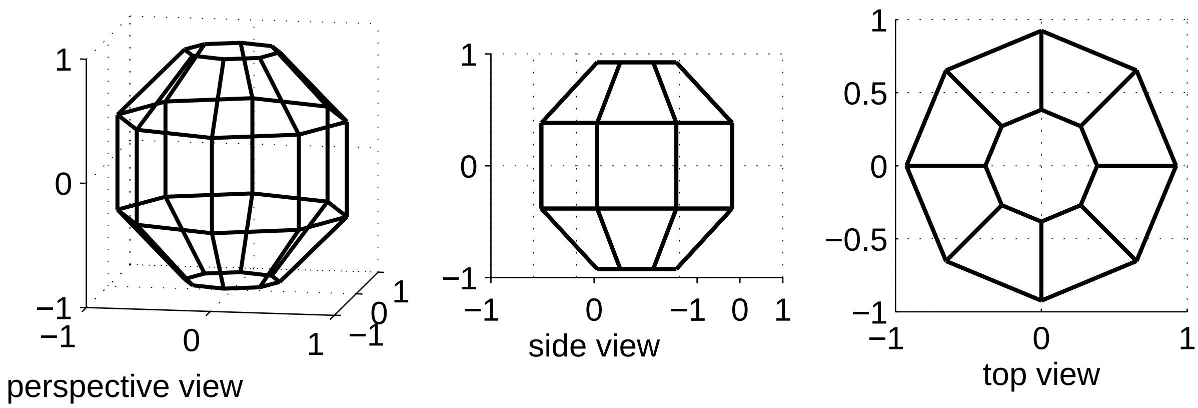

Figure 1: “OctaGlobe” polyhedron shown in (a) perspective, (b) side view (polar regions at top and bottom), and (c) top view (tropical regions around perimeter).

The polyhedron can be unfolded to form a flat series of connected rectangles and trapezoids (Fig. 2) as well as the polar octagons shown in Fig. 1c. In order to produce the appropriate projection, we must find the relationship between the height A and width B of the low-latitude rectangles and the height C and high-latitude width D of the mid-latitude trapezoids (Fig. 2). The width D also tells us the size of the high-latitude octagons. Readers who do not wish to review the derivation can skip to the end of this section for the values of these distances.

In order to give an approximate representation of a sphere, the polyhedron's vertices touch the surface of a sphere. The vertices occur at latitude θ of $\pm 22.5^o$ and $\pm 67.5^o$, and at

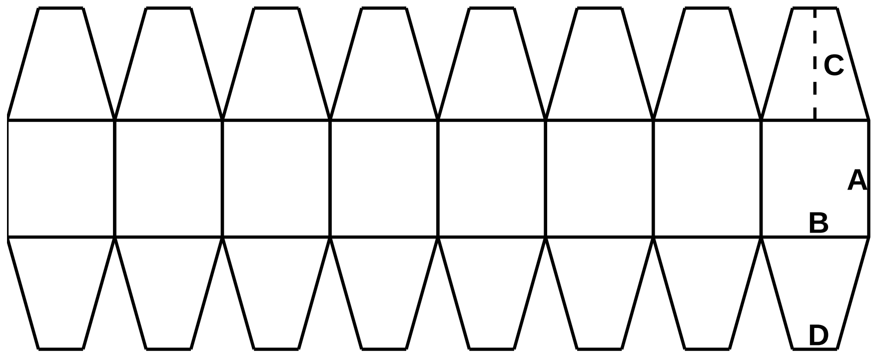

Figure 2: “Unfolded” OctaGlobe as it would appear if tropical and mid-latitude faces were laid flat.

longitudes ϕ of $0^o, 45^o, 90^o, ..., 315^o$. We can calculate the relationship between the radius R of the sphere and A, B, C, D by considering vectors $\vec{x}_n$ connecting the center of the sphere to various vertices. These lengths represent the magnitudes of the difference of several $\vec{x}_n$. We compute the magnitudes by writing the vectors in cartesian coordinates, using

$$x = R(\cos\theta)\cos\phi \tag{1a}$$
$$y = R(\cos\theta)\sin\phi \tag{1b}$$
$$z = R\sin\theta, \tag{1c}$$

where (x, y, z) are the Cartesian coordinates for the vector going from the center of the sphere to latitude and longitude (θ, ϕ). The distance A represents the magnitude of the difference between $\vec{x}_1$, the vertex at $(\theta_1, \phi_1) = (-22.5^o, 0)$ and $\vec{x}_2$, the vertex at $(\theta_2, \phi_2) = (22.5^o, 0)$. Individual Cartesian coordinates for $\vec{x}_2 - \vec{x}_1$, according to (1), are given by

$$x_2 - x_1 = R[\cos\theta_2 - \cos\theta_2] = 0 \tag{2a}$$
$$y_2 - y_1 = 0 \tag{2b}$$
$$z_1 - z_2 = 2R\sin\theta_1 \tag{2c}$$

For B, the appropriate vectors are $\vec{x}_2$ and $\vec{x}_3$, the vertex at $(\theta_3, \phi_3) = (22.5^o, 45^o)$, with Cartesian values

$$x_3 - x_2 = R\cos\theta_2[\cos\phi_3 - \cos\phi_1] \tag{3a}$$
$$y_3 - y_2 = R\cos\theta_2[\sin\phi_3] \tag{3b}$$
$$z_3 - z_2 = 0 \tag{3c}$$

where we use the fact that $\phi_2 = \phi_1$. The distance D is given by the difference between $\vec{x}_4$ at $(\theta_4, \phi_4) = (67.5^o, 0^o)$ and $\vec{x}_5$ at $(\theta_5, \phi_5) = (67.5^o, 45^o)$, with

$$\begin{aligned} x_5 - x_4 &= R\cos\theta_4[\cos\phi_3 - \cos\phi_1] && (4a) \\ y_5 - y_4 &= R\cos\theta_4[\sin\phi_3 - \sin\phi_1] && (4b) \\ z_5 - z_4 &= 0. && (4c) \end{aligned}$$

The calculation for C is slightly more complicated, because it represents the distance not between vertices but between the midpoints of pairs of vertices. The midpoints are $\vec{x}_6$ between $(22.5^o, 0^o)$ and $(22.5^o, 45^o)$, and $\vec{x}_7$ between $(67.5^o, 0^o)$ and $(67.5^o, 45^o)$. The corresponding Cartesian coordinates are

$$\begin{aligned} x_7 - x_6 &= \frac{1}{2}[R\cos\theta_4 - R\cos\theta_2][\cos\phi_1 + \cos\phi_3] && (5a) \\ y_7 - y_6 &= \frac{1}{2}[R\cos\theta_4 - R\cos\theta_2][\sin\phi_1 + \sin\phi_3] && (5b) \\ z_7 - z_6 &= R\sin\theta_4 - R\sin\theta_1. && (5c) \end{aligned}$$

Given the Cartesian differences $(\Delta x, \Delta y, \Delta z)$, the magnitudes are $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. Using the expressions above for individual Cartesian components, we get

$$\begin{aligned} A &= 2R\sin\theta_2 && (6a) \\ B &= R\cos\theta_2\sqrt{\sin^2\phi_3 + (1 - \sin\phi_3)^2} && (6b) \\ C &= R\sqrt{\frac{1}{4}(\cos\theta_4 - \cos\theta_2)^2[(1 + \cos\phi_3)^2 + \sin^2\phi_3] + (\sin\theta_4 - \sin\theta_2)^2} && (6c) \\ D &= R\cos\theta_4\sqrt{2\sin^2\phi_3 + 1 - 2\sin\phi_3} && (6d) \end{aligned}$$

where we also use the fact that $\phi_1 = 0$.

While computers can easily calculate the numerical values of the sines and cosines in the preceding formulas, it is worth noting that these all involve special angles. Thus we have $\cos\phi_3 = \sin\phi_3 = 1/\sqrt{2}$. Since $\theta_2 = \phi_3/2$, we can use the half angle formulas,

$$\sin\alpha = \sqrt{(1 - \cos 2\alpha)/2} \quad (7a)$$

$$\cos\alpha = \sqrt{(1 + \cos 2\alpha)/2}, \quad (7b)$$

which give

$$\sin\theta_2 = \sqrt{(1 - 1/\sqrt{2})/2} \quad (8a)$$

$$\cos\theta_2 = \sqrt{(1 + 1/\sqrt{2})/2}. \quad (8b)$$

Finally, since $\theta_4 = 90^o - \theta_2$, $\sin\theta_4 = \cos\theta_2$ and $\cos\theta_4 = \sin\theta_2$.

Inserting the numerical values above into the distance formulas (6), after some manipulation we obtain,

$$A/R = \sqrt{\sqrt{2}(\sqrt{2}-1)} \approx .7654 \tag{9a}$$
$$B/R = 1/\sqrt{2} \approx .7071 \tag{9b}$$
$$C/R = \sqrt{5/4 - 1/\sqrt{2}} \approx .7368 \tag{9c}$$
$$D/R = \sqrt{3/2 - \sqrt{2}} \approx .2929. \tag{9d}$$

Our calculations show that $A \neq B$ ($B/A \approx .9626$), even though they represent a sector that is 45^o wide in both latitude and longitude. This reflects the fact that at $\pm 22.5^o$, the east-west distance for a small change in longitude is slightly smaller than the north-south distance for the same change in latitude.

2 Projection on to OctaGlobe

Having established the dimensions of the OctaGlobe, we can now write the formulas for projecting a latitude-longitude location (θ, ϕ) on to the flat surface in Fig. 2. The projection is relatively simple because the faces of the OctaGlobe align with latitude and longitude.

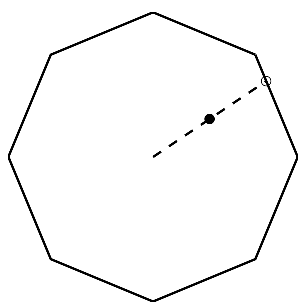

Figure 3: Polar octagon illustrating projection from latitude-longitude coordinates.

For the tropical and mid-latitude faces, we map the latitude linearly on to the vertical distance from the centerline ($y = 0$) of the figure, with the equator at $y = 0$, $\theta = \pm 22.5^o$ at $y = \pm A/2$, and $\theta = \pm 67.5^o$ at $y \pm (A/2 + C)$. Similarly, each 45^o longitude sector is mapped linearly into the horizontal distance across the rectangle (equatorward of 22.5^o) or trapezoid (poleward of 22.5^o). For the polar octagons, we first find the location along the perimeter of the octagon that corresponds to the longitude (Fig. 3, point marked by small open circle). The distance along the perimeter corresponds to the latitude. The mapped point will fall somewhere along the line between this point and the center of the octagon (Fig. 3, dashed line). The distance along the line is proportional to the difference in latitude from the pole (Fig. 3, small filled circle making a given latitude), with a latitude of 67.5^o falling on the perimeter.

For (latitude, longitude) (θ, ϕ), the projection formulas are as follows. We define (X, Y) as distances in the (horizontal, vertical) directions in Fig. 2. For $|\theta| < 22.5^o$,

$$X = B(\phi/45^o) \tag{10a}$$
$$Y = A(\theta/45^o). \tag{10b}$$

For $22.5^o < |\theta| \leq 67.5^o$, we first define ϕ_0, the longitude $22.5^o + (45^o)n$ (where n is an integer) closest to ϕ. Then $\Delta\phi = \phi - \phi_0$. The projection is given by

$$X = B(\phi_0/45^o) + [B - (B - D)(|\theta| - 22.5^o)/45^o](\Delta\phi/45^o) \tag{11a}$$
$$Y = \mathrm{sgn}(\theta)(A/2 + C(|\theta| - 22.5^o)/45^o) \tag{11b}$$

For $|\theta| > 67.5^o$, we define

$$r = (1 + \sqrt{2})D(90^o - |\theta|/45^o)/\cos(|\Delta\phi|) \tag{12}$$

The location of the point within each octagon, using Cartesian coordinates (X_O, Y_O) centered on the pole, is

$$X_O = r\cos\phi_P \tag{13a}$$
$$Y_O = r\sin\phi_P \tag{13b}$$

where ϕ_P is given by

$$\phi_P = (\phi - 157.5^o) \quad (\theta > 0) \tag{14a}$$
$$\phi_P = (-\phi - 22.5^o) \quad (\theta < 0). \tag{14b}$$

Another possible projection would be to use the line intersecting the center of the sphere and a given (latitude, longitude) location on the sphere. The intersection of this line with the OctaGlobe would give the projection on to the OctaGlobe faces. The formula for such a projection is somewhat more complicated than what we have described above. Since the faces of the Octaglobe are fairly close to the surface of the sphere, the difference in location between the two methods is small and so here we use the simpler formula.