Weak Convergence Results for Multiple Generations of a Branching Process

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Abstract

We establish limit theorems involving weak convergence of multiple generations of critical and supercritical branching processes. These results arise naturally when dealing with the joint asymptotic behavior of functionals defined in terms of several generations of such processes. Applications of our main result include a functional central limit theorem (CLT), a Darling-Erdös result, and an extremal process result. The limiting process for our functional CLT is an infinite dimensional Brownian motion with sample paths in the infinite product space $(C_0[0,1])^\infty$, with the product topology, or in Banach subspaces of $(C_0[0,1])^\infty$ determined by norms related to the distribution of the population size of the branching process. As an application of this CLT we obtain a central limit theorem for ratios of weighted sums of generations of a branching processes, and also to various maximums of these generations. The Darling-Erdös result and the application to extremal distributions also include infinite dimensional limit laws. Some branching process examples where the CLT fails are also included.

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1 Introduction

Our interest in limit theorems for multiple generations of branching processes is motivated by both practical and theoretical considerations. The practical side stems from the use of branching processes to model certain aspects of scientific experiments. One such problem area is Polymerase Chain Reaction (PCR) experiments. In such an experiment, an initial amount of DNA is amplified for use in various biological experiments. The PCR experiment evolves in three phases; an exponential phase, a linear phase, and a pleateau phase, with branching processes and their variants frequently used to model the exponential phase. One of the goals of such experiments is to "quantitate" the initial number of DNA molecules in a sample or equivalently, estimate the number of ancestors in a branching process [5]. The statistical estimate of the initial number of ancestors is a function of the estimate of the mean of the branching process [5], and in order to make this estimate, data are used from the last few cycles (generations) at the end of the exponential phase. Since the cycle(generation) corresponding to the end of exponential phase is somewhat arbitrary, it is natural to consider the joint distributions of the generations involved to determine whether two different scientists with different choices for the end of the exponential phase obtain consistent results. Furthermore, these joint distributions can also be used to estimate the end of the exponential phase.

Theoretical motivation for our results involves the desire to understand analogues of classical functional limit theorems for i.i.d. sequences that hold for multiple generations of the stochastic processes arising in the branching setting. What we present here deals with weak convergence results. Theorem 1 is our main result, and allows a large number of applications, a few of which are presented explicitly as Applications 1-3, and Theorems 2 and 3 in Section 2. Application 1 is a functional CLT, yielding a Donsker type result, Application 2 a Darling-Erdös result, and Application 3 an extremal process result, all obtained under best possible conditions. For example, in the functional CLT we use only second moments, and in the Darling-Erdös result we use the moment condition shown in [3] to be necessary for this result for i.i.d. sequences. A similar comment applies to the application to extremal processes. Here the regularly varying tail condition assumed is precis ely that required for the limiting maximal distribution at $t = 1$ to exist for an i.i.d sequence. Other applications are also possible once one has Theorem 1 available, but in Theorems 2 and 3 we turn to some applications of our functional CLT. Theorem 2 yields a strengthening of the functional CLT to the Banach spaces $c_{0,\lambda}(C_0[0,1])$. Another consequence of Application 1 is a new proof of the CLT for the non-parametric maximum likelihood estimate of the mean of a supercritical branching process. A previous proof of this in [9] involves a martingale CLT, whereas the proof herein is an elegant application of our functional CLT result with $t = 1$, and the asymptotic independence obtained in the coordinates of the limiting process. Moreover, our proof allows us to extend this result to allow the application of a broad range of weights on the various generations. In [9] all the weights are equal to one.

In order to describe our results in more detail we begin with a brief description of the branching process. Let $\{\xi_{n,j}, j \geq 1, n \geq 1\}$ denote a double array of integer valued i.i.d. random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, and having probability distribution $\{p_j : j \geq 0\}$, i.e. $P(\xi_{1,1} = k) = p_k$. Then $\{Z_n : n \geq 0\}$ denotes the Galton-Watson process initiated by a single ancestor $Z_0 \equiv 1$. It is iteratively defined on $(\Omega, \mathcal{F}, P)$ for $n \geq 1$ by

$$Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{n,j}.$$

Let $m = E(Z_1)$. It is well known that if $m > 1$ (*i.e.* the process is supercritical), then $Z_n \to \infty$ with positive probability and that the probability that the process becomes extinct, namely q, is less than one. The complement of the set $\cup_{n=1}^{\infty}\{Z_n = 0\}$ is the so called survival set, and is denoted by S. If $m > 1$, then $P(S) = 1 - q$ and $Z_n \to \infty$ a.s. on S. Also, $q = 0$ if and only if $p_0 = 0$. If $m \leq 1$, then assuming $p_1 \neq 1$ when $m = 1$, the process becomes extinct with probability one, i.e. $P(S) = 0$. To avoid degenerate situations we assume throughout the paper, without further mention, that $p_0 + p_1 < 1$.

The paper is organized as follows: Section 2 develops the basic notation and states the main results of the paper. Section 3 contains the proof of Theorem 1, and Sections 4 and 5 that of the CLT applications in Theorem's 2 and 3, respectively. Section 6 contains examples providing some insight into the CLT for subcritical processes, and also for supercritical processes when one uses deterministic normalizations. In this latter example one does not get a Gaussian limit law, but a certain mixture of Gaussian laws. This mixture can be anticipated from the Kesten-Stigum result, but its precise expression requires some interesting analysis. In particular, these examples show precisely why the random normalizations used in our theorems are possibly the "best choice" if one wants classical results to persist in limit theorems for multiple generations of these processes.

2 Notation and Main Results

In this section we state the main result of the paper. This result allows us to obtain a wide variety of limit theorems for branching processes based on $r(n)$-generations, where $1 \leq r(n) \leq n$. Following its statement we present some interesting consequences and applications. In particular, in these applications the integer sequence $\{r(n)\}$ may approach infinity as n goes to infinity. As will be seen, they all follow rather immediately from our main result when combined with various classical limit theorems for i.i.d. sequences.

Throughout (M, d) is a complete separable metric space with distance d, and M^∞ denotes the infinite product of copies of M with the product topology, metrized by

$$d_\infty(\mathbf{x}, \mathbf{y}) = \sum_{k \geq 1} \frac{1}{2^k} \frac{d(x_k, y_k)}{1 + d(x_k, y_k)}. \tag{2.1}$$

In our applications M is the real line or some function space. If M is the real line, then the distance is the usual one, and for our functional CLT application M denotes the set of all continuous functions on [0,1] that vanish at 0, which we denote by $C_0[0,1]$. Of course, then $C_0[0,1]$ is a Banach space in the supremum norm

$$q(f) = \sup_{0 \leq t \leq 1} |f(t)|, \tag{2.2}$$

and the distance used is $d(f, g) = q(f - g)$, $f, g \in C_0[0,1]$. Application 3 below contains a different choice of M, and others are certainly possible, but these suffice to provide a sampling of possible consequences of our main theorem.

Since we want to study the asymptotic behavior of $r(n)$ generations of the branching process, and $r(n)$ may well converge to infinity, it is useful for these purposes to define

$$\mathbf{X}_{n,r(n)} \equiv (X_{n,Z_{n-1}}, X_{n-1,Z_{n-2}}, \cdots X_{n-r(n)+1,Z_{n-r(n)}}, z, z, \cdots), \tag{2.3}$$

where z is a fixed element in M,

$$X_{n-j+1,Z_{n-j}} = H_{Z_{n-j}}(\xi_{n-j+1,1}, \cdots, \xi_{n-j+1,Z_{n-j}}), \tag{2.4}$$

and the mappings $H_k(\cdot)$ take R^k into M are Borel measurable. Hence $\mathbf{X}_{n,r(n)}$ is an element of the infinite product space M^∞. Moreover, in our applications M always contains a zero element which we denote by 0, and if we take the fixed element $z \in M$ in (2.3) to be this 0, then we have

$$\mathbf{X}_{n,r(n)} \equiv (X_{n,Z_{n-1}}, X_{n-1,Z_{n-2}}, \cdots X_{n-r(n)+1,Z_{n-r(n)}}, 0, 0, \cdots). \tag{2.5}$$

We will use $\Rightarrow$ to denote weak convergence of probability measures. Our main theorem for the random vectors $\mathbf{X}_{n,r(n)}$ is the following.

Theorem 1. *Let $m \geq 1$, assume $1 \leq r(n) \leq n$ with $\lim_{n\to\infty} r(n) = \infty$, and that $\mathbf{X}_{n,r(n)}$ is defined as in (2.3)-(2.4). Also assume that if $\{\xi_j : j \geq 1\}$ are i.i.d. non-negative integer valued random variable with $\mathcal{L}(\xi_1) = \mathcal{L}(Z_1)$, then the M-valued random elements $\{H_k : k \geq 1\}$ used to define $\mathbf{X}_{n,r(n)}$ are such that*

$$H_k(\xi, \cdots, \xi_k) \Rightarrow H \tag{2.6}$$

on (M, d). Then the probability measures

$$\mu_n = \mathcal{L}(\mathbf{X}_{n,r(n)} | Z_{n-1} > 0) \tag{2.7}$$

converge weakly on (M^∞, d_∞), i.e. we have

$$\mu_n \Rightarrow \mathcal{L}(B_1, B_2, \cdots), \tag{2.8}$$

where the B_i's are independent copies of H.

Next we present three immediate applications of Theorem 1. They include a functional CLT, a Darling-Erdös Theorem, and also an extremal process result. It is interesting to observe that the limiting distributions of the coordinates of $\mathbf{X}_{n,r(n)}$ are asymptotically independent, whereas the generations of the branching process itself are correlated.

Application 1: Let $m \geq 1$, $0 < \sigma^2 = E(Z_1^2) < \infty$, and assume $1 \leq r(n) \leq n$ with $\lim_{n\to\infty} r(n) = \infty$. Take $M = C_0[0,1]$ with the sup-norm q, define $H_k(\xi, \cdots, \xi_k)(0) = 0$, and for $0 \leq t \leq 1$ set

$$H_k(\xi, \cdots, \xi_k)(t) = \frac{1}{\sigma\sqrt{k}} \sum_{i=1}^{\lfloor tk \rfloor} (\xi_i - m) + (tk - \lfloor tk \rfloor) \frac{1}{\sigma\sqrt{k}} (\xi_{\lfloor tk \rfloor + 1} - m). \tag{2.9}$$

Then Donsker's Invariance principle implies (2.6) holds with $\mathcal{L}(H)$ the probability measure induced on $C_0[0,1]$ by a standard Brownian motion starting at zero when $t = 0$. If $\mathbf{X}_{n,r(n)}$ is defined as in

(2.3-5) with H_k as in (2.9), and $(C_0[0,1])^\infty$ has the product topology induced when using the norm q on $C_0[0,1]$, then an immediate consequence of Theorem 1 is that the probability measures

$$\mu_n = \mathcal{L}(\mathbf{X}_{n,r(n)}|Z_{n-1} > 0) \tag{2.10}$$

converge weakly there, i.e. we have

$$\mu_n \Rightarrow \mathcal{L}(B_1, B_2, \cdots), \tag{2.11}$$

where the B_i's are independent Brownian motions.

Application 2: Let $m \geq 1$, $0 < \sigma^2 = E(Z_1^2) < \infty$, $\lim_{t\to\infty} LLtE(Z_1^2 I(|Z_1| \geq t)) = 0$, where $Lt = \log_e(t \vee e)$ and $LLt = L(Lt)$. In addition, assume $1 \leq r(n) \leq n$ with $\lim_{n\to\infty} r(n) = \infty$, and take $M = R^1$. Define

$$H_k(\xi, \cdots, \xi_k) = a_k \max_{1\leq j\leq k} \frac{\sum_{i=1}^{j}(\xi_i - m)}{\sigma\sqrt{j}} - b_k, \tag{2.12}$$

where $a_k = (2LLk)^{\frac{1}{2}}$ and $b_k = 2LLk + \frac{1}{2}LLLk - \frac{1}{2}L(4\pi)$. Then the Darling-Erdös Theorem as in Theorem 2 of [3] implies (2.6) holds with $\mathcal{L}(H)$ the probability measure induced on M by the cumulative distribution function

$$G(x) = \exp\{-e^{-x}\}, \ -\infty, x < \infty. \tag{2.13}$$

If $\mathbf{X}_{n,r(n)}$ is defined as in (2.3-5) with H_k as in (2.12), and $M^\infty = R^\infty$ has the product topology , then an immediate consequence of Theorem 1 is that the probability measures

$$\mu_n = \mathcal{L}(\mathbf{X}_{n,r(n)}|Z_{n-1} > 0) \tag{2.14}$$

converge weakly there, and we have

$$\mu_n \Rightarrow \mathcal{L}(B_1, B_2, \cdots), \tag{2.15}$$

where the B_i's are independent random variables with cumulative distribution function $G(x)$ given by (2.13).

Application 3: Let $m \geq 1$, assume $F(x) = P(Z_1 \leq x) < 1$ for all $x \in R^1$, and that $1 - F(x)$ is regularly varying at ∞ with exponent $-\alpha$ where $\alpha > 1$. In addition, assume $1 \leq r(n) \leq n$ with $\lim_{n\to\infty} r(n) = \infty$. Then by Theorem 6.3, p.455 of [4], there exists $a_j > 0$ such that

$$\lim_{j\to\infty} P(\frac{1}{a_j}\max\{0, \xi_1, \cdots, \xi_j\} \leq x) = \exp\{-x^{-\alpha}\}, \ x > 0, \tag{2.16}$$

and zero for $x \leq 0$. Now define the extremal process $m_k(t)$ which is zero in $[0, \frac{1}{n})$ and

$$m_k(t) = \frac{1}{a_k}\max\{0, \xi_1, \cdots, \xi_j\}, \ \frac{j}{n} \leq t < \frac{j+1}{k}, \ j = 1, \cdots, k-1, \tag{2.17}$$

and

$$m_k(t) = \frac{1}{a_k}\max\{0, \xi_1, \cdots, \xi_k\}, \ t \geq 1. \tag{2.18}$$

Let M denote the finite, non-decreasing functions $z(t)$on $[0,\infty)$ such that $z(0)=0$ and $z(t)=z(1)$ for $t\geq 1$.Then M is a complete separable metric space in the Levy metric d_L on M, and if

$$H_k(\xi_1,\cdots,\xi_k)(t)=m_k(t),\ 0\leq t<\infty, \tag{2.19}$$

then by Theorem 2.1 and 3.1 of [7] we have (2.6) where $\{H(t):0\leq t<\infty\}$ is a Markov extremal process with sample paths in M. Therefore, if $\mathbf{X}_{n,r(n)}$ is defined as in (2.3-5) with H_k as in (2.19), and M^∞ has the product topology, then an immediate consequence of Theorem 1 is that the probability measures

$$\mu_n=\mathcal{L}(\mathbf{X}_{n,r(n)}|Z_{n-1}>0) \tag{2.20}$$

converge weakly there, and we have

$$\mu_n\Rightarrow\mathcal{L}(B_1,B_2,\cdots), \tag{2.21}$$

where the B_i's are independent copies of the Markov process $\{H(t):0\leq t<\infty\}$.

As is easily seen, Theorem 1 combined with other classical limit theorems for i.i.d sequences provides many possible limit theorems for suitable choices of the random elements $\mathbf{X}_{n,r(n)}$. However, what we turn to next are some applications and extensions of the functional CLT of application one. The first involves a functional CLT in Banach subspaces of $(C_0[0,1])^\infty$ determined by weighted analogues of the q-norm. That is, let $\lambda=\{\lambda_j:j\geq 1\}$ be a sequence of strictly positive numbers, and for $\mathbf{f}=(f_1,f_2,\cdots)\in(C_0[0,1])^\infty$ define

$$q_\lambda(\mathbf{f})=\sup_{j\geq 1}\lambda_j||f_j||, \tag{2.22}$$

where $||\cdot||$ is the supremum norm on $C_0[0,1]$. Also, let $c_{0,\lambda}(C_0[0,1])$ be the subspace of $(C_0[0,1])^\infty$ given by

$$c_{0,\lambda}(C_0[0,1])=\{\mathbf{f}=(f_1,f_2,\cdots)\in(C_0[0,1])^\infty:\lim_{j\to\infty}\lambda_j||f_j||=0\}. \tag{2.23}$$

Then $q_\lambda(\mathbf{f})$ is a norm making the subspace $c_{0,\lambda}(C_0[0,1])$ a Banach space.

As before we will use $\Rightarrow$ to denote weak convergence of probability measures. Our functional central limit theorem in $c_{0,\lambda}(C_0[0,1])$ is the following.

Theorem 2. *Let $m\geq 1$ and assume $1\leq r(n)\leq n$ with $\lim_{n\to\infty}r(n)=\infty$. Also assume that the offspring distribution $\mathcal{L}(\xi)=\mathcal{L}(Z_1)$ is such that $0<\sigma^2=E(\xi^2)<\infty$ and satisfies one of the following conditions:*

$$P(|\xi-m|\geq x)\leq\beta e^{-\theta x^2},\ \text{for all } x\geq 0,\ \text{or} \tag{2.24}$$

$$E(|\xi-m|^\rho)<\infty \text{ for some } \rho\geq 2, \tag{2.25}$$

and that $r(n)=o(n)$. Let $\mathbf{X}_{n,r(n)}$ be defined as in (2.3-5) with $H_k(\xi_1,\cdots,\xi_k)$ as in (2.9). If (2.24) holds and we take $\lambda=\{\lambda_j\}$ where $\lambda_j=(\delta_j\log(j+3))^{-\frac{1}{2}}$ and $\lim_{j\to\infty}\delta_j=\infty$, then on the Banach space $c_{0,\lambda}(C_0[0,1])$ the probability measures

$$\mu_n=\mathcal{L}(\mathbf{X}_{n,r(n)}|Z_{n-1}>0). \tag{2.26}$$

are such that

$$\mu_n \Rightarrow \mathcal{L}(B_1, B_2, \cdots), \tag{2.27}$$

where the B_i's are independent standard Brownian motions. If instead we assume (2.25) and $\lambda = \{\lambda_j\}$ where $\lambda_j = j^{-\frac{(1+\delta)}{\rho}}$ and $\delta > 0$, then we again have (2.27) on $c_{0,\lambda}(C_0[0,1])$.

If G is Gaussian random variable with mean zero and variance one, then for all $x \geq 0$

$$P(\sup_{0 \leq t \leq 1} B(t) \leq x) = P(|G| \leq x).$$

Hence Theorem 2 and the continuous mapping theorem applied to the processes $\{\mathbf{X}_{n,r(n)}(\cdot) : n \geq 1\}$ with values in $c_{0,\lambda}(C_0[0,1])$ immediately imply the following result.

Corollary 1. *If (2.24) or (2.25) holds with corresponding $\{\lambda_j : j \geq 1\}$ as indicated, then the conditions of Theorem 2 imply that*

$$\lim_{n \to \infty} P(\max_{1 \leq j \leq r(n)} \lambda_j \frac{(Z_{n-j+1} - mZ_{n-j})}{\sigma Z_{n-j}^{\frac{1}{2}}} \leq x) = P(\sup_{j \geq 1} \lambda_j G_j \leq x),$$

where $G_1, G_2, \cdots$ are i.i.d. $N(0,1)$ random variables. In addition, we also have

$$\lim_{n \to \infty} P(\max_{1 \leq j \leq r(n)} \lambda_j (\frac{\max_{1 \leq k \leq Z_{n-j}} (0 \vee \sum_{i=1}^{k} (\xi_{n-j+1,i} - m))}{\sigma Z_{n-j}^{\frac{1}{2}}}) \leq x) = P(\sup_{j \geq 1} \lambda_j |G_j| \leq x).$$

Remark 1. *If one wants results similar to those of the corollary with $\lambda_j = 1$, then Theorem 2, or application one applies, as long as we restrict the maximum to be over only finitely many j's, say $j \in \{1, 2, \cdots, d\}$.*

In Theorem 3 below we obtain a CLT for ratios of weighted sums of a supercritical branching process $\{Z_n : n \geq 1\}$. When the weights are all one the result appeared in [9] using a martingale CLT for the proof. Our proof is completely different. It uses Application 1 in an important way and allows the ratios to consist of weighted sums. We begin with some notation.

Throughout we assume $\{b_j : j \geq 1\}$ is a sequence of non-negative numbers with $b_1 > 0$, and $\sup_{j \geq 1} b_j < \infty$. Now set

$$X_n = (\frac{N_n}{D_n} - m)\sqrt{D_n},$$

where

$$N_n = b_1 Z_n + b_2 Z_{n-1} + \cdots + b_n Z_1,$$

$$D_n = b_1 Z_{n-1} + b_2 Z_{n-2} + \cdots + b_n Z_0,$$

and we understand X_n to be zero if $D_n = 0$. Then we have the following CLT.

Theorem 3. *Let $m > 1,\ 0 < \sigma^2 \equiv E((Z_1 - m)^2) < \infty$, and assume $\{b_j : j \geq 1\}$ is a sequence of non-negative numbers with $b_1 > 0$ and $\sup_{j \geq 1} b_j < \infty$. For $k \geq 2$, let*

$$\theta_k = b_k(b_1 m^{k-1} + \cdots + b_{k-1} m + b_k + \sum_{j=k+1}^{\infty} b_j \frac{1}{m^{j-k}})^{-1}, \tag{2.28}$$

and when $k = 1$ set

$$\theta_1 = b_1(b_1 + \sum_{j=2}^{\infty} b_j \frac{1}{m^{j-1}})^{-1}. \tag{2.29}$$

If

$$\Lambda^2 = \sum_{j=1}^{\infty} \theta_j b_j \sigma^2,$$

then $\Lambda^2 < \infty$, and for all real x we have

$$\lim_{n \to \infty} P(X_n \leq x | S) = P(G \leq x), \tag{2.30}$$

where G is a mean zero Gaussian random variable with $E(G^2) = \Lambda^2$.

We also have the following corollary.

Corollary 2. *If $b_1 = b_2 = \cdots = b_d = 1$ and $b_j = 0$ for $j \geq d+1$, or $b_j = 1$ for all $j \geq 1$, then in both situations*

$$\lim_{n \to \infty} P(X_n \leq x | S) = P(G \leq x), \tag{2.31}$$

where G is a mean zero Gaussian random variable with $E(G^2) = \sigma^2$.

Remark 2. *If we condition on $\{Z_{n-1} > 0\}$ instead of S in Theorem 3, or its Corollary, the limit is the same.*

3 Proof of Theorem 1

The proof of Theorem 1 is based on a lemma for weak convergence in infinite product spaces, and an iterative technique developed in Lemma 2 below.

Let (M, d) be a complete separable metric space, μ a Borel probability measure on (M, d), and $\pi : M \to M$ Borel measurable. Define,

$$\mu^{\pi}(A) = \mu(\pi^{-1}(A))$$

for all Borel sets A of (M, d). Let M^{∞} denote the infinite product space with the product topology and typical point $\mathbf{s} = (s_1, s_2, \cdots)$. If z is a fixed point in M, we define the mapping $\pi_l : M^{\infty} \to M^{\infty}$, for $l \geq 1$, by

$$\pi_l(\mathbf{s}) = (s_1, s_2, \cdots, s_l, z, z, \cdots,).$$

We now indicate a lemma concerning weak convergence in product spaces. Its proof is easily anticipated.

Lemma 1. *Let M be as above and assume $\{\mu_n : n \geq 1\}$ and μ_∞ are Borel probability measures on M^∞ with the product topology. Then $\{\mu_n : n \geq 1\}$ converges weakly to μ_∞ if and only if $\mu_n^{\pi_l}$ converges weakly to $\mu_\infty^{\pi_l}$ for all $l \geq 1$.*

The results of the next lemma are used several times in the proof of Theorem 1, and hence we combine them in an easily accessed form, When $m = 1$ the result follows from Theorem 2, p. 20, of [1], and when $m > 1$ a minor modification of Theorem 3 of [1], p. 41, suffices.

Lemma 2. *Let $\{Z_n : n \geq 0\}$ be a Galton-Watson process with $Z_0 = 1$. If $m = 1$, then for each $J \in [1, \infty)$*

$$\lim_{n\to\infty} \frac{P(1 \leq Z_n \leq J)}{P(Z_n > 0)} = 0. \tag{3.1}$$

If $m > 1$, then there exists a constant $\gamma \in (0, 1)$ such that

$$\lim_{n\to\infty} P(Z_n = k)/\gamma^n = \nu_k, \tag{3.2}$$

where $0 \leq \nu_k < \infty$ for all $k \geq 1$.

Proof of (2.8) Let μ denote the probability measure induced by H on M, and μ_∞ be the infinite product measure formed by μ on M^∞. Also let μ_n denote the law of $\mathbf{X}_{n,r(n)}$ when Z_{n-1} is conditioned to be stricty positive,i.e. for A a Borel subset of M^∞ we have

$$\mu_n(A) = P(\mathbf{X}_{n,r(n)} \in A | Z_{n-1} > 0).$$

By Lemma 1 it is sufficient to establish, for each $l \geq 1$, the weak convergence of $\mu_n^{\pi_l}$ to $\mu_\infty^{\pi_l}$. If we identify the range space of π_l with M^l in the obvious way, then it suffices to show that on M^l we have that

$$\lambda_n = \mathcal{L}(X_{n,Z_{n-1}}, X_{n-1,Z_{n-2}}, \cdots, X_{n-l+1,Z_{n-l}} | Z_{n-1} > 0)$$

converges weakly to $(\mu)^l$, the l-fold product of μ on that space.

To establish weak convergence of λ_n to $(\mu)^l$, it is sufficient by Theorem 2.2 of [2] to show for arbitrary continuity sets E_i of the measure μ on M that

$$\lim_{n\to\infty} \lambda_n(E_1 \times E_2 \times \cdots \times E_l) = \prod_{j=1}^{l} \mu(E_j). \tag{3.3}$$

We will now verify

Lemma 3. *If $m \geq 1$ and $r(n) \to \infty$, then (3.3) holds.*

Proof Set

$$\theta_n = \lambda_n(\prod_{j=1}^{l} E_j).$$

Then,

$$\theta_n = E(\prod_{j=1}^{l} I_{A_{n,j}})/(P(Z_{n-1} > 0)),$$

where

$$A_{n,j} = \{X_{n-j+1,Z_{n-j}} \in E_j,\ Z_{n-j} > 0\}$$

for $1 \le j \le l$. Let $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_n = \sigma(\{\xi_{k,j} : j \ge 1\} : 1 \le k \le n)$ for $n \ge 1$. Also, to simplify the notation, write $A_{n,j} = A_j$, for $1 \le j \le l$. Therefore,

$$\theta_n P(Z_{n-1} > 0) = E(E(\prod_{j=1}^{l} I_{A_j} | \mathcal{F}_{n-1})) = E(E(I_{A_1}|\mathcal{F}_{n-1}) \prod_{j=2}^{l} I_{A_j}).$$

Setting $\beta_n \equiv E(I_{A_1}|\mathcal{F}_{n-1})$, and writing

$$\beta_n = \Delta_n + \mu(E_1),$$

where $\Delta_n = E(I_{A_1}|\mathcal{F}_{n-1}) - \mu(E_1)$ we have

$$\theta_n P(Z_{n-1} > 0) = \mu(E_1) E(\prod_{j=2}^{l} (I_{A_j}) + e_n, \tag{3.4}$$

where

$$e_n = E(\prod_{j=2}^{l} I_{A_j} \Delta_n).$$

We will now show that $\lim_{n\to\infty} e_n / P(Z_{n-1} > 0) = 0$. To this end, let $\epsilon > 0$ and note that

$$\limsup_{n\to\infty} e_n / P(Z_{n-1} > 0) \le \limsup_{n\to\infty} E(|\Delta_n| I_{Z_{n-2}>0}) / P(Z_{n-1} > 0) \le I + II,$$

where

$$I = \limsup_{n\to\infty} E(|P(A_1|\mathcal{F}_{n-1}) - \mu(E_1)| I_{(Z_{n-1}>0,\ Z_{n-2}>0)}) / P(Z_{n-1} > 0),$$

and

$$II = \limsup_{n\to\infty} E(|P(A_1|\mathcal{F}_{n-1}) - \mu(E_1)| I_{(Z_{n-1}=0,\ Z_{n-2}>0)}) / P(Z_{n-1} > 0).$$

Using the Markov property and branching property of $\{Z_n\}$ we have

$$P(A_1 | Z_{n-1} = j) = P(X_{n,Z_{n-1}} \in E_1 | Z_{n-1} = j) = P(H_j(\xi_1, \cdots, \xi_j) \in E_1).$$

Hence, given $\epsilon > 0$, and that E_1 is a μ-continuity set, (2.6) implies

$$|P(H_j(\xi_1, \cdots, \xi_j) \in E_1) - \mu(E_1)| \le \epsilon$$

for $j \ge j_0(\epsilon, E_1)$ independent of n. Therefore,

$$\begin{aligned} I &= \limsup_{n\to\infty} E(|P(A_1|Z_{n-1}) - \mu(E_1)| I_{(Z_{n-1}>0\ \ Z_{n-2}>0)}) / P(Z_{n-1} > 0) && (3.5)\\ &\le \limsup_{n\to\infty} (\Sigma_{1,n} + \Sigma_{2,n}) / P(Z_{n-1} > 0), && (3.6) \end{aligned}$$

where

$$\Sigma_{1,n} = \sum_{1 \le j_1 \le j_0} \sum_{j_2 \ge 1} |P(A_1 | Z_{n-1} = j_1) - \mu(E_1)| P(Z_{n-1} = j_1, Z_{n-2} = j_2),$$

$$\Sigma_{2,n} = \sum_{j_1 \ge j_0+1} \sum_{j_2 \ge 1} |P(A_1 | Z_{n-1} = j_1) - \mu(E_1)| P(Z_{n-1} = j_1, Z_{n-2} = j_2).$$

Thus,

$$I \leq \limsup_{n\to\infty}[2P(1 \leq Z_{n-1} \leq j_0)/P(Z_{n-1} > 0) + \epsilon] = \epsilon, \tag{3.7}$$

since $\lim_{n\to\infty} P(1 \leq Z_{n-1} \leq j_0)/P(Z_{n-1} > 0) = 0$ is immediate by Lemma 2 if $m \geq 1$, i.e recall $P(Z_n > 0) \to 1 - q > 0$ when $m > 1$.

As for II, observe that

$$\begin{aligned} II &\leq 2\limsup_{n\to\infty}\sum_{j\geq 1} P(Z_{n-2} = j, Z_{n-1} = 0)/P(Z_{n-1} > 0) && (3.8)\\ &= 2\limsup_{n\to\infty}\sum_{j\geq 1} p_0^j P(Z_{n-2} = j)/P(Z_{n-1} > 0) && (3.9)\\ &= 2\limsup_{n\to\infty}[f_{n-2}(p_0) - f_{n-2}(0)]/P(Z_{n-1} > 0), && (3.10)\end{aligned}$$

where $f_{n-2}(\cdot)$ is the generating function of Z_{n-2}. Thus for $m > 1$ we have $II = 0$ since $f_{n-2}(p_0)$ and $f_{n-2}(0)$ both converge to q and $P(Z_n > 0) \to 1 - q > 0$ when $m > 1$. If $m = 1$, then since $0 < \sigma^2 < \infty$ by assumption in Theorem 1, we have by [1], p19, for all $k \geq 1$ that

$$\lim_{n\to\infty} P(Z_{n-k} > 0)/P(Z_n > 0) = 1. \tag{3.11}$$

In addition, we have

$$[f_{n-2}(p_0) - f_{n-2}(0)] \leq \sum_{k=1}^{k_0} P(Z_{n-2} = k) + p_0^{k+1} P(Z_{n-2} \geq k_0 + 1).$$

Thus by Lemma 2 and that k_0 can be taken arbitrarily large we see

$$\limsup_{n\to\infty}[f_{n-2}(p_0) - f_{n-2}(0)]/P(Z_{n-2} > 0) = 0. \tag{3.12}$$

Therefore, when $m = 1$, we also see $II = 0$.

Since $e_n \geq 0$, the conditions of Theorem 1 imply

$$\lim_{n\to\infty} e_n/P(Z_{n-1} > 0) = 0$$

for all $m \geq 1$.

Now returning to (3.4) and iterating we get,

$$\theta_n P(Z_{n-1} > 0) = \prod_{j=1}^{l-1} \mu(E_j)E(I_{A_l}) + \sum_{j=0}^{l-2} \mu(E_j)e_{n-j}, \tag{3.13}$$

where $E_0 = M$ and $e_{n-j} = E(\prod_{k=2+j}^{l} I_{A_k}\Delta_{n-k})$. Furthermore, using an argument similar to the one used to prove $e_n/P(Z_{n-1} > 0) \to 0$, one can also show that

$$\lim_{n\to\infty} e_{n-j}/P(Z_{n-1} > 0) = 0.$$

Finally, note that

$$
\begin{aligned}
E(I_{A_l}) &= P(X_{n-l+1,Z_{n-l}} \in E_l, Z_{n-l} > 0) && (3.14)\\
&= \sum_{j\geq 1} P(H_j(\xi_1,\cdots,\xi_j) \in E_l) P(Z_{n-l} = j) && (3.15)\\
&= I_n + II_n + III_n, && (3.16)
\end{aligned}
$$

where

$$
I_n = \sum_{j=1}^{j_0(\epsilon,E_l)} P(H_j(\xi_1,\cdots,\xi_j) \in E_l) P(Z_{n-l} = j). \tag{3.17}
$$

$$
II_n = \sum_{j>j_0(\epsilon,E_l)} (P(H_j(\xi_1,\cdots,\xi_j) \in E_l) - \mu(E_l)) P(Z_{n-l} = j), \tag{3.18}
$$

and

$$
III_n = \mu(E_l) P(Z_{n-l} > j_0(\epsilon, E_l)); \tag{3.19}
$$

and $j_0(\epsilon, E_l)$ is such that for all $j \geq j_0(\epsilon, E_l)$

$$
|P(H_j(\xi_1,\cdots,\xi_j) \in E_l) - \mu(E_l)| < \epsilon. \tag{3.20}
$$

The existence of $j_0(\epsilon, E_l)$ follows from (2.6), and that E_l is a continuity set of μ.

Using (3.11), which holds for $m \geq 1$, we have by Lemma 2 that

$$
\lim_{n\to\infty} P(1 \leq Z_{n-l} \leq j_0(\epsilon, E_l))/P(Z_{n-1} > 0) = 0,
$$

and hence it follows that $I_n/P(Z_{n-1} > 0) \to 0$ as $n \to \infty$. Furthermore, using (3.20) and that here we also have $P(Z_{n-l} > j_0(\epsilon, E_l))/P(Z_{n-1} > 0) \to 1$, it follows that

$$
\limsup_{n\to\infty} II_n/P(Z_{n-1} > 0) \leq \epsilon.
$$

Finally, similar ideas imply $III_n/P(Z_{n-1} > 0) \to \mu(E_l)$. Thus, $\lim_{n\to\infty} E(I_{A_l})/P(Z_{n-1} > 0) = \mu(E_l)$, and hence (3.13) implies $\lim_{n\to\infty} \theta_n = \prod_{j=1}^{l} \mu(E_j)$. Thus (3.3) holds and Theorem 1 is proven. ∎

4 Proof of Theorem 2

Application 1 of Theorem 1 implies (2.27) on $(C_0[0,1])^\infty$ with the product topology. Now we turn to its proof for the spaces $c_{0,\lambda}(C_0[0,1])$ and their stated norms q_λ. Given that weak convergence in the product topology implies the finite dimensional distributions of any finite set of coordinates converges in correct fashion, it suffices to show the probability measures of (2.26) are tight on the spaces $c_{0,\lambda}(C_0[0,1])$. This is the content of our next lemma. Its proof establishes Theorem 2.

Lemma 4. *Let $\{\mu_n : n \geq 1\}$ be as in (2.26), assume $m \geq 1$, and that $r(n) \to \infty$. If (2.24) holds and $\lambda_j = (\delta_j \log(j+3))^{-\frac{1}{2}}$, where $\lim_{j\to\infty} \delta_j = \infty$, then the $\{\mu_n : n \geq 1\}$ are tight on $c_{0,\lambda}(C_0[0,1])$. Similarly, if (2.25) holds and $\lambda_j = j^{-\frac{(1+\delta)}{\rho}}$ for $\delta > 0$, then we also have $\{\mu_n : n \geq 1\}$ tight on $c_{0,\lambda}(C_0[0,1])$.*

Proof. Since the finite dimensional distributions of any finite set of coordinates of $\{\mu_n\}$ converge weakly to the corresponding ones for $\mathcal{L}(B_1, B_2, \cdots)$, standard arguments allow us to finish the proof by showing the $\{\mu_n\}$ are tight on $c_{0,\lambda}(C_0[0,1])$).

To establish tightness we apply the remark in [8], p. 49. To show this remark applies we use the fact that the distributions of any finite set of coordinates are tight (since they are convergent), and therefore it suffices to show for each $\epsilon > 0$ that there exists a $d(\epsilon)$ such that $d \geq d(\epsilon)$ implies

$$\limsup_{n\to\infty} P(q_\lambda(Q_d(\mathbf{X}_{n,r(n)})) \geq \epsilon | Z_{n-1} > 0) \leq \epsilon. \tag{4.1}$$

Here $Q_d(\mathbf{f}) = (0, \cdots, 0, f_{d+1}, f_{d+2}, \cdots)$ for $\mathbf{f} \in (C_0[0,1])^\infty$. Since we are assuming $r(n)$ tends to infinity, for all n sufficiently large we have

$$P(q_\lambda(Q_d(\mathbf{X}_{n,r(n)})) \geq \epsilon | Z_{n-1} > 0) \leq \sum_{j=d+1}^{r(n)} I_{n,j},$$

where

$$I_{n,j} \equiv P(\max_{1\leq l\leq Z_{n-j}} |\sum_{k=1}^{l} (\xi_{n-j+1,k} - m)| \geq Z_{n-j}^{\frac{1}{2}} \epsilon \lambda_j^{-1} | Z_{n-1} > 0).$$

Setting $J_{n,j} = I_{n,j} P(Z_{n-1} > 0)$ we see

$$J_{n,j} \quad = \quad \sum_{r=1}^{\infty} P(\max_{1\leq l\leq r} |\sum_{k=1}^{l} (\xi_{n-j+1,k} - m)| \geq r^{\frac{1}{2}} \epsilon \lambda_j^{-1}, Z_{n-j} = r, Z_{n-1} > 0). \tag{4.2}$$

Thus

$$J_{n,j} \leq \sum_{r=1}^{\infty} P(\max_{1\leq l\leq r} |\sum_{k=1}^{l} (\xi_{n-j+1,k} - m)| \geq r^{\frac{1}{2}} \epsilon \lambda_j^{-1} | Z_{n-j} = r) P(Z_{n-j} = r), \tag{4.3}$$

and by the branching property we see

$$J_{n,j} \leq \sum_{r=1}^{\infty} P(\max_{1\leq l\leq r} |\sum_{k=1}^{l} (\xi_k - m)| \geq r^{\frac{1}{2}} \epsilon \lambda_j^{-1}) P(Z_{n-j} = r), \tag{4.4}$$

where $\{\xi_k : k \geq 1\}$ are i.i.d. with law that of the offspring distribution. Since $\lambda_j \to \infty$ there exists a $j_0 = j_0(\epsilon)$ such that $j \geq j_0$ and Ottavianni's inequality implies

$$J_{n,j} \leq 2 \sum_{r=1}^{\infty} P(|\sum_{k=1}^{r} (\xi_k - m)| \geq \frac{r^{\frac{1}{2}} \epsilon \lambda_j^{-1}}{2}) P(Z_{n-j} = r). \tag{4.5}$$

Now under (2.24), Lemma 4.1 of [6] implies for all $r \geq 1, j \geq 1$ that

$$P(|\sum_{k=1}^{r} (\xi_k - m)| \geq \frac{r^{\frac{1}{2}} \epsilon \lambda_j^{-1}}{2}) \leq 2\beta \exp\{-\theta \epsilon^2 \lambda_j^{-2}/4\} = 2\beta (j+3)^{-\frac{\theta \epsilon^2 \delta_j^2}{4}},$$

and hence for $j \geq j_0$ we have

$$I_{n,j} \leq 4\beta(j+3)^{-\frac{\theta\epsilon^2\delta_j^2}{4}} P(Z_{n-j} > 0)/P(Z_{n-1} > 0). \tag{4.6}$$

When $m = 1$ and $0 < Var(Z_1) = \sigma^2 < \infty$, we have by Theorem 1, p.19, of [1] that $\lim_{n\to\infty} nP(Z_n > 0) = 2/\sigma^2$. Hence for $n - j \geq n_0$ we have

$$\frac{P(Z_{n-j} > 0)}{P(Z_{n-1} > 0)} = \frac{(n-j)P(Z_{n-j} > 0)}{(n-1)P(Z_{n-1} > 0)} \frac{(n-1)}{(n-j)} \leq 2\frac{(1-\frac{1}{n})}{(1-\frac{j}{n})} \leq \frac{2}{(1-\frac{j}{n})}. \tag{4.7}$$

Thus for $j = o(n), j \geq j_0$, we have

$$I_{n,j} \leq 16\beta(j+3)^{-\frac{\theta\epsilon^2\delta_j^2}{4}}. \tag{4.8}$$

Now take $j_1 = j_1(\epsilon)$ such that $j \geq j_1$ implies $\theta\epsilon^2\delta_j^2 > 2$. Given $\epsilon > 0, r(n) = o(n)$ and $d > d_0(\epsilon) \equiv \max(j_0, j_1, \frac{16\beta}{\epsilon} + 1)$, we have

$$\limsup_{n\to\infty} P(q_\lambda(Q_d(\mathbf{X}_{n,r(n)})) \geq \epsilon | Z_{n-1} > 0) \leq \limsup_{n\to\infty} \sum_{j=d+1}^{r(n)} I_{n,j} \leq \epsilon.$$

Hence the lemma is proven under (2.24) if $m = 1$. If $m > 1$, then (4.8) is an even easier consequence of (4.6) since $\lim_{n\to\infty} P(Z_n > 0) = 1 - q > 0$. Hence if $r(n) = o(n)$, the lemma also holds in this case.

If (2.25) holds, then for all $r > 0$ and $\rho \geq 2$ we have a constant $B_\rho < \infty$ such that an application of Markov's inequality and Corollary 8.2 in [4], p.151, implies

$$P(|\sum_{k=1}^{r}(\xi_k - m)| \geq \frac{r^{\frac{1}{2}}\epsilon\lambda_j^{-1}}{2}) \leq B_\rho \frac{(E(|\xi_1 - m|^\rho)}{(\frac{\epsilon\lambda_j^{-1}}{2})^\rho}.$$

Hence the arguments can be completed as before, since under (2.25) we have $\lambda_j^{-1} = j^{\frac{(1+\delta)}{\rho}}$. Thus the lemma is proven. ∎

5 Proof of Theorem 3

Before we turn to the proof of Theorem 3 we provide a brief lemma, and recall that if $m > 1$ and $0 < \sigma^2 \equiv E((Z_1 - m)^2) < \infty$, then the Kesten-Stigum theorem, [1], p. 20, implies that with probability one that

$$\lim_{n\to\infty} W_n = W, \tag{5.1}$$

where $W_n = \frac{Z_n}{m^n}$, and $W > 0$ almost surely on the survival set S.

Lemma 5. *Under the given assumptions, we have almost surely that*

$$\lim_{n\to\infty} \frac{D_n}{m^n} = \sum_{j\geq 1} \frac{b_j}{m^j} W. \tag{5.2}$$

Furthermore, for $k \geq 1$ almost surely on S we have

$$\lim_{n\to\infty} \frac{b_k Z_{n-k}}{D_n} = \theta_k, \tag{5.3}$$

where θ_k is given as in (2.28) and (2.29).

Proof. Observe that

$$\frac{D_n}{m^n} = \sum_{k=1}^{n} \frac{b_k Z_{n-k}}{m^n} = \sum_{k=1}^{n} \frac{b_k}{m^k} W + \sum_{k=1}^{n} \frac{b_k}{m^k} (\frac{Z_{n-k}}{m^{n-k}} - W),$$

and since $\lim_{n\to\infty} Z_n/m^n = W$ almost everywhere by (5.1), an elementary argument easily implies $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{b_k}{m^k}(\frac{Z_{n-k}}{m^{n-k}} - W) = 0$ almost everywhere. Thus (5.2) holds. Combining (5.2) and (5.1) with $W > 0$ almost surely on S, we thus have (5.3). Hence the lemma is proven. ∎

For the proof of Theorem 3 recall that if $D_n = 0$, then we understand X_n to be zero. Furthermore, if $D_n > 0$, we then have

$$X_n = \sum_{j=1}^{n} \sqrt{\frac{b_j Z_{n-j}}{D_n}} \sqrt{b_j Z_{n-j}} (\frac{Z_{n-j+1}}{Z_{n-j}} - m),$$

and for $1 \le d \le n$ we define

$$X_{n,d} = \sum_{j=1}^{d} \sqrt{\frac{b_j Z_{n-j}}{D_n}} \sqrt{b_j Z_{n-j}} (\frac{Z_{n-j+1}}{Z_{n-j}} - m).$$

Of course, when $D_n = 0$, we understand X_n and $X_{n,d}$ as given in these formulas to be zero. We also use

$$\tilde{X}_n = \sqrt{\frac{D_n}{m^n}} X_n \text{ and } \tilde{X}_{n,d} = \sqrt{\frac{D_n}{m^n}} X_{n,d},$$

and their formulas analogous to those above for X_n and $X_{n,d}$.

Proof of Theorem 3. Take $\epsilon > 0$, and to simplify the notation set $\gamma_n = (\frac{D_n}{m^n})^{\frac{1}{2}}$. Then

$$P(X_n \le x|S) = P(\tilde{X}_n \le x\gamma_n|S), \tag{5.4}$$

$$P(\tilde{X}_n \le x\gamma_n|S) \le P(\tilde{X}_{n,d} \le (x+\epsilon)\gamma_n|S) + P(|\tilde{X}_n - \tilde{X}_{n,d}| \ge \epsilon\gamma_n|S), \tag{5.5}$$

and

$$P(|\tilde{X}_n - \tilde{X}_{n,d}| \ge \epsilon\gamma_n|S) \le P(|\tilde{X}_n - \tilde{X}_{n,d}| \ge \epsilon\delta|S) + P(0 < \gamma_n < \delta|S). \tag{5.6}$$

Since $\epsilon > 0$ is given, we choose $\delta > 0$ sufficiently small that $P(0 < W < 2\delta^2)/P(S) < \epsilon$. Since $\lim_{n\to\infty} \gamma_n = W^{\frac{1}{2}} > 0$ almost surely on S, there exists $n_0 = n_0(\delta)$ such that $n \ge n_0$ implies

$$P(0 < \gamma_n < \delta|S) < \epsilon. \tag{5.7}$$

Once $\epsilon, \delta > 0$ are fixed, we choose $d_0 = d_0(\epsilon, \delta)$ such that $d \ge d_0$ implies that uniformly in n

$$P(|\tilde{X}_n - \tilde{X}_{n,d}| \ge \epsilon\delta|S) \le \epsilon. \tag{5.8}$$

To obtain d_0 we observe that $P(|\tilde{X}_n - \tilde{X}_{n,d}| \ge \epsilon\delta|S) \le P(|\tilde{X}_n - \tilde{X}_{n,d}| \ge \epsilon\delta)/P(S)$, and since the branching property easily implies

$$E((\tilde{X}_n - \tilde{X}_{n,d})^2) = \sum_{j=d+1}^{n} m^{-n} b_j E(Z_{n-j-1})\sigma^2 = \sum_{j=d+1}^{n} \sigma^2 b_j m^{-(j+1)},$$

we have

$$E((\tilde{X}_n - \tilde{X}_{n,d})^2) \le \frac{\sigma^2 M m^{-(d+1)}}{(m-1)}, \tag{5.9}$$

where $M = \sup_{j\ge 1} b_j < \infty$. Hence Markov's inequality, (5.9), and the above reasoning allows us to choose d_0 independent of n, so (5.8) holds.

Since $P(X_{n,d} \le x+\epsilon|S) = P(\tilde{X}_{n,d} \le (x+\epsilon)\gamma_n|S)$, by combining (5.4)-(5.8) we have for $d \ge d_0$ that

$$P(X_n \le x|S) \le P(X_{n,d} \le x+\epsilon|S) + 2\epsilon. \tag{5.10}$$

Similarly, we also have for $d \ge d_0$ that

$$P(X_n \le x|S) \ge P(X_{n,d} \le x-\epsilon|S) - 2\epsilon. \tag{5.11}$$

Now let

$$X'_{n,d} = \sum_{j=1}^{d} \sqrt{\theta_j}\sqrt{b_j Z_{n-j}}\Big(\frac{Z_{n-j+1}}{Z_{n-j}} - m\Big), \tag{5.12}$$

and observe that by setting $t = 1$ in the functional CLT of Application 1, the continuous mapping theorem immediately implies the uniform stochastic boundedness of

$$\{\sqrt{b_j Z_{n-j}}\Big(\frac{Z_{n-j+1}}{Z_{n-j}} - m\Big) : 1 \le j \le d, n \ge 1\}$$

when these variables are conditioned on the event $\{Z_{n-1} > 0\}$. Therefore, for each fixed d we have from (5.3) of Lemma 5 and the previously mentioned uniform stochastic boundedness that

$$\lim_{n\to\infty} P(|X_{n,d} - X'_{n,d}| \ge \epsilon|S) = 0. \tag{5.13}$$

In addition, by the CLT provided by Application 1 we easily have

$$\lim_{n\to\infty} P(X'_{n,d} \le x) = P(G_d \le x) \tag{5.14}$$

for all real x, where G_d is a mean zero Gaussian random variable with variance $\Lambda_d^2 = \sum_{j=1}^d \theta_j b_j \sigma^2$.

Combining (5.10), (5.13), and (5.14) we have for all $d \ge d_0$ and all real x that

$$\limsup_{n\to\infty} P(X_n \le x|S) \le P(G_d \le x+2\epsilon) + 3\epsilon. \tag{5.15}$$

Using (5.11), a similar argument implies for all $d \ge d_0$ and all real x that

$$\liminf_{n\to\infty} P(X_n \le x|S) \ge P(G_d \le x-2\epsilon) - 3\epsilon. \tag{5.16}$$

Now take $d_1 = d_1(\epsilon)$ sufficiently large such that $d \ge d_1$ implies for all real x that

$$P(G_d \le x) - \epsilon \le P(G \le x) \le P(G_d \le x) + \epsilon, \tag{5.17}$$

where G is as in the proposition. This condition follows easily since $\Lambda_d^2 \to \Lambda^2 < \infty$.

Letting d tend to infinity in (5.15) and (5.16), (5,17) implies for all x that

$$\limsup_{n\to\infty} P(X_n \le x|S) \le P(G \le x+2\epsilon) + 3\epsilon, \tag{5.18}$$

and

$$\liminf_{n\to\infty} P(X_n \le x|S) \ge P(G \le x - 2\epsilon) - 3\epsilon \tag{5.19}$$

Letting $\epsilon \downarrow 0$ in (5.18) and (5.19), the theorem is proven. ■

Proof of Corollary 2. Given the postulated b_j, (2.28), (2.29), and simple calculation implies

$$\theta_j = \frac{(m-1)m^{d-j}}{(m^d-1)},\ j = 1, \cdots, d,$$

or

$$\theta_j = (m-1)m^{-j},\ j \ge 1.$$

Hence the conclusion of Theorem 3 implies (2.31) with $E(G^2) = \sum_{j=1}^d \theta_j \sigma^2 = \sigma^2$. ■

6 Examples

In this section we provide some examples where the CLT fails. We focus on the CLT as it is perhaps the result one might expect would be most likely to persist under suitable modifications of our basic assumptions. In the first example failure results from our branching process $\{Z_n : n \ge 0\}$ being subcritical. Hence, even though one has the same conditional independence structure as in the critical and supercritical cases, its behavior is quite different. In the other example the CLT fails through the use of deterministic normalizers.

Subcritical Branching Fails the CLT: Our result concerns the limit of

$$\mathcal{L}(Z_{n-1}^{\frac{1}{2}}(\frac{Z_n}{Z_{n-1}} - m)|Z_{n-1} > 0),$$

and shows that even for this single distribution the CLT always fails. This is easy to see since the distribution of all the $\bar{H}_k$′s of the following lemma are discrete.

Lemma 6. *Assume that* $E(Z_1^2) < \infty$ *and set* $L_n = Z_{n-1}^{\frac{1}{2}}(\frac{Z_n}{Z_{n-1}} - m)$. *Then, for any* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P(L_n \le x|Z_{n-1} > 0) = \sum_{k\ge 1} P(\sqrt{k}\bar{H}_k \le x)\theta_k, \tag{6.1}$$

where $\bar{H}_k = \frac{1}{k}\sum_{i=1}^k(\xi_i - m)$, $\{\xi_i : i \ge 1\}$ *are i.i.d. with* $\mathcal{L}(\xi_1) = \mathcal{L}(Z_1)$, *and* $\{\theta_k : k \ge 1\}$ *is a probability distribution.*

Proof of Lemma 6. Let $x \in \mathbb{R}$. Then, by the branching property we easily have

$$\begin{aligned} P(L_n \le x|Z_{n-1} > 0) &= \sum_{k\ge 0} P(L_n \le x; Z_{n-1} = k|Z_{n-1} > 0) & (6.2)\\ &= \sum_{k\ge 1} P(\sqrt{k}\bar{H}_k \le x)P(Z_{n-1} = k|Z_{n-1} > 0). & (6.3)\end{aligned}$$

Since $m < 1$, Yaglom's Theorem on p.18 of [1] implies that $\lim_{n\to\infty} P(Z_{n-1} = k|Z_{n-1} > 0) = \theta_k$, where $\{\theta_k : k \ge 1\}$ is a probability distribution. Thus, by the generalized dominated convergence theorem, it follows that

$$\lim_{n\to\infty} P(L_n \le x|Z_n > 0) = \sum_{k\ge 1} P(\sqrt{k}\bar{H}_k \le x)\theta_k. \tag{6.4}$$

This completes the proof of the lemma. $\blacksquare$

Deterministic Normalizers Prevent the CLT: Even when $m > 1$, the next example shows the spatial finite dimensional distributions related to Application 1 fail to be Gaussian when we use canonical deterministic normalizations $m^{\frac{n-1}{2}}$ instead of $Z_{n-1}^{\frac{1}{2}}$ in our CLT results. Of course, the motivation for these normalizations results from the Kesten-Stigum result, see (5.1), and in this situation the limit laws are a mixture of Gaussian laws and the random variable W that appears in that result.

Proposition 1. *Let $m > 1$, $0 < \sigma^2 \equiv E((\xi_{1,1} - m)^2) < \infty$. For $i = 1, \cdots, l$, set $H_{n-i+1} = \frac{m^{\frac{(n-i)}{2}}}{\sigma}(\frac{Z_{n-i+1}}{Z_{n-i}} - m)$ when $Z_{n-j} > 0$, and zero otherwise, and let*

$$B_{n,i} = \{H_{n-i+1} \leq t_i\}, \tag{6.5}$$

where $t_1, \cdots, t_l \in (-\infty, \infty)$. Then

$$\lim_{n\to\infty} P(B_{n,1} \cap \cdots \cap B_{n,l} \cap \{Z_{n-1} > 0\}) = E(\Phi(t_1 W^{1/2}) \cdots \Phi(t_l W^{1/2}) I_{S_0}), \tag{6.6}$$

where $S_0 \equiv \{\lim_{n\to\infty} \frac{Z_n}{Z_{n-1}} = m\}$ and W is as in (5.1).

Proof of Proposition 1. For $Z_{n-1} > 0$ set $G_n = \frac{Z_{n-1}^{\frac{1}{2}}}{\sigma}(\frac{Z_n}{Z_{n-1}} - m)$, and when $Z_{n-1} = 0$ define $G_n = 0$. If $\Phi(\cdot)$ is the standard Gaussian cumulative distribution function, then under the assumptions of the proposition for $t \in (-\infty, \infty)$, we have almost surely that

$$\lim_{n\to\infty} P(G_n \leq t | Z_{n-1}) = \Phi(t) I_{S_0} + I_{\{0 \leq t\}} I_{S_0'}. \tag{6.7}$$

Furthermore, we have almost surely on S_0 that

$$\lim_{n\to\infty} P(G_n \leq t W_{n-1}^{1/2} | Z_{n-1}) = \Phi(t W^{1/2}) I_{S_0}, \tag{6.8}$$

and on S' that

$$\lim_{n\to\infty} P(G_n \leq t W_{n-1}^{1/2} | Z_{n-1}) = I_{S'}. \tag{6.9}$$

To verify (6.7-9) we observe that by the branching property

$$P(G_n \leq t | Z_{n-1}) = h(Z_{n-1}, t),$$

where $h(k,t) = P(\sum_{i=1}^{k} (\xi_i - m)/k^{1/2} \leq t)$ and $\{\xi_i : i \geq 1\}$ are i.i.d random variables independent of the branching process with law that of Z_1. Since $m > 1$, on the set S_0 we have $Z_{n-1} \to \infty$, and hence by the classical CLT on S_0 we have

$$\lim_{n\to\infty} P(G_n \leq t | Z_{n-1}) = \lim_{n\to\infty} h(Z_{n-1}, t) = \Phi(t) I_{S_0}. \tag{6.10}$$

On the the complement of the survival set, namely $S' = \cup_{n=1}^{\infty}\{Z_n = 0\}$, we eventually have for n sufficiently large that $G_n = 0$. Thus on S' eventually in n we have

$$P(G_n \leq t|Z_{n-1}) = P(0 \leq t)I_{S'} = I_{\{0\leq t\}}I_{S'}.$$

Thus (6.7) holds as it is known that $P(S\Delta S_0) = 0$.

On S_0, from (6.10) and that $\Phi(\cdot)$ is continuous, we have uniform convergence in t as $n \to \infty$. Therefore, on S_0, as n $\to \infty$ we have

$$|P(G_n \leq tW_{n-1}^{1/2}|Z_{n-1}) - \Phi(tW_n^{1/2})| = o(1), \tag{6.11}$$

where as usual o(1) means the term goes to zero. Since $\Phi(\cdot)$ is continuous, as $n \to \infty$, (5.1) implies with probability one that

$$|\Phi(tW^{1/2}) - \Phi(tW_n^{1/2})| = o(1), \tag{6.12}$$

Hence (6.8) holds.

On S' we eventually have as n gets large that $Z_{n-1} = 0$, and hence $G_n = W_{n-1} = 0$. Therefore, on S' eventually in n we have

$$P(G_n \leq tW_{n-1}^{1/2}|Z_{n-1}) = P(0 \leq 0)I_{S'},$$

and (6.9) follows.

For $i = 1, \cdots, l$ we set

$$A_{n,i} = \{G_{n-i+1} \leq t_i W_{n-i}^{1/2}\}, \tag{6.13}$$

where $t_1, \cdots, t_l \in (-\infty, \infty)$. Then we will show

$$\lim_{n\to\infty} P(A_{n,1} \cap \cdots \cap A_{n,l}) = E(\Phi(t_1W^{1/2}) \cdots \Phi(t_lW^{1/2})I_{S_0}) + P(S_0'), \tag{6.14}$$

and that

$$\lim_{n\to\infty} P(A_{n,1} \cap \cdots \cap A_{n,l} \cap \{Z_{n-1} > 0\}) = E(\Phi(t_1W^{1/2}) \cdots \Phi(t_lW^{1/2})I_{S_0}). \tag{6.15}$$

Since the sets $A_{n,i} = B_{n,i}$, we thus have (6.6) from (6.15), and the proposition will be proven.

Arguing as above, we see that $\lim_{n\to\infty} I_{A_{n,i}\cap S'} = I_{S'}$ almost surely. Since $P(S'\Delta S_0') = 0$, the bounded convergence theorem then implies

$$\lim_{n\to\infty} P(A_{n,1} \cap \cdots \cap A_{n,l} \cap S_0') = P(S_0').$$

Hence it suffices to show that

$$\lim_{n\to\infty} P(A_{n,1} \cap \cdots \cap A_{n,l} \cap S_0) = E(\Phi(t_1W^{1/2}) \cdots \Phi(t_lW^{1/2})I_{S_0}). \tag{6.16}$$

Since $\lim_{n\to\infty} E(|I_S - I_{\{Z_{n-k}>0\}}|) = 0$ for each integer k, the bounded convergence theorem easily implies (6.16) provided we show

$$\lim_{n\to\infty} P(A_{n,1} \cap \cdots \cap A_{n,l} \cap \{Z_{n-1} > 0\}) = E(\Phi(t_1W^{1/2}) \cdots \Phi(t_lW^{1/2})I_{S_0}). \tag{6.17}$$

Now

$$P(A_{n,1} \cap \cdots \cap A_{n,l} \cap \{Z_{n-1} > 0\}) = E(I_{A_{n,2}} \cdots I_{A_{n,l}} I_{\{Z_{n-1}>0\}} E(I_{A_{n,1}} | \mathcal{F}_{n-1})) \tag{6.18}$$

and since $\lim_{n\to\infty} E(|I_{S_0} - I_{\{Z_{n-k}\}}|) = 0$ for each integer k, the bounded convergence theorem and (6.8) together easily imply that as n tends to infinity

$$P(A_{n,1} \cap \cdots \cap A_{n,l} \cap \{Z_{n-1} > 0\}) = E(I_{A_{n,2}} \cdots I_{A_{n,l}} I_{S_0} \Phi(t_1 W^{1/2})) + o(1). \tag{6.19}$$

By the bounded convergence theorem, that $\lim_{n\to\infty} E(|I_{S_0} - I_{\{Z_{n-2}>0\}}|) = 0$, and

$$\lim_{n\to\infty} \Phi(t_1 W_{n-2}^{1/2}) = \Phi(t W^{1/2}),$$

we have as n tends to infinity that

$$E(I_{A_{n,2}} \cdots I_{A_{n,l}} I_{S_0} \Phi(t_1 W^{1/2})) = E(I_{A_{n,2}} \cdots I_{A_{n,l}} I_{\{Z_{n-2}>0\}} \Phi(t_1 W_{n-2}^{1/2})) + o(1). \tag{6.20}$$

Now

$$\begin{aligned} &E(I_{A_{n,2}} \cdots I_{A_{n,l}} I_{\{Z_{n-2}>0\}} \Phi(t_1 W_{n-2}^{1/2})) \\ &\qquad = E(I_{A_{n,3}} \cdots I_{A_{n,l}} I_{\{Z_{n-2}>0\}} \Phi(t_1 W_{n-2}^{1/2}) E(I_{A_{n,2}} | \mathcal{F}_{n-2})), \end{aligned}$$

and by the Markov property and (6.8) we can iterate the previous steps to show

$$\begin{aligned} &E(I_{A_{n,3}} \cdots I_{A_{n,l}} I_{\{Z_{n-2}>0\}} \Phi(t_1 W_{n-2}^{1/2}) E(I_{A_{n,2}} | \mathcal{F}_{n-2})) \\ &\qquad = E(I_{A_{n,3}} \cdots I_{A_{n,l}} I_{\{Z_{n-3}>0\}} \Phi(t_1 W_{n-3}^{1/2}) \Phi(t_2 W_{n-3}^{1/2})) + o(1) \end{aligned}$$

as n tends to infinity. Continuing in this way, using the bounded convergence theorem to modify things as we go along, we eventually obtain (6.17) and the proposition is proved. ■

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