Path Properties of Multigenerational Samples from Branching Processes

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Abstract

This paper is concerned with the study of functional limit theorems constructed using data from multiple generations of a supercritical branching process. These results arise naturally when samples are obtained from several successive generations of a supercritical branching process, and one is interested in the joint asymptotic behavior of various statistical functionals constructed from the data. More precisely, let $\{Z_n : n \geq 0\}$ be a supercritical branching process and set $R_n = Z_{n-1}^{-1} Z_n$. Also let, $\mathbf{R}_{n,r(n)} = (R_n, R_{n-1}, \cdots, R_{n-r(n)+1})$. Since the number of generations sampled may increase without limit, to formulate our results it is natural to embed $\mathbf{R}_{n,r(n)}$ in R^∞, and its related functional forms in infinite products of continuous function spaces. The limit theorems we consider include various forms of the functional law of large numbers (consistency) and also a functional central limit theorem (asymptotic normality) under minimal moment conditions. The limiting process in our functional limit theorem is an infinite dimensional Brownian motion in the infinite product space $(C_0[0,1])^\infty$, with the product topology. In order to study rates of convergence in these results, we also include related infinite dimensional functional laws of the iterated logarithm of Strassen and Chung-Wichura type in the space $(C_0[0,1])^\infty$. Connections to various statistical issues involving PCR and other related data are discussed.

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1 Introduction

The primary focus of this paper is the study of path properties of various functionals constructed from data sampled successively from multiple generations of a supercritical branching process initiated by a single ancestor at time 0. The functionals that we study include ratios of generation sizes, the maxima of partial sums of the n^{th} generation population, and natural functional forms related to such quantities. A number of factors motivated this study. First, in several scientific experiments, samples are obtained from the n^{th} generation to perform inferences on the mean and variance parameter of a branching process. The choice of n is somewhat adhoc and varies between scientists and lab technicians. One of the goals of this paper is to study, from an asymptotic perspective, the joint behavior of such estimates, when one samples from successive generations. Second, from a probabilistic perspective, we wanted to understand the analogues of classical functional limit theorems for the stochastic processes involving multiple generations of supercritical branching processes that arise in this setting.

An example of the first motivation arises in the study of Polymerase Chain Reaction (PCR) experiments. In such an experiment, an initial amount of DNA is amplified for use in various biological experiments. The PCR experiment evolves in three phases; an exponential phase, a linear phase, and a pleateau phase. Branching processes and their variants have been used to model data from PCR experiments during the exponential phase ([17], pp. 231). The mean of the branching process is related to the quantity called the *efficiency* of the PCR. One of the goals of the PCR experiments is to "quantitate" the initial number of DNA molecules in a sample or equivalently, estimate the number of ancestors in a branching process ([20]). In an end point assay, data are obtained from the last two cycles (generations) corresponding to the end of the exponential phase, and these are used to estimate the mean of the branching process. The statistical estimate of the initial number of ancestors is a function of the estimate of the mean of the branching process ([20]). Since the cycle(generation) corresponding to the end of exponential phase is somewhat arbitrary and varies betweens labs and the technicians involved, a natural question is if two different technicians with different choices for the end of the exponential phase obtain consistent results for the same experiment. The results of this paper help answer this question in the affirmative in the sense that our results imply joint convergence of a broad array of multigenerational samples from the branching process. For instance, our central limit theorem enables an experimenter to construct confidence regions for the "mean vector" using the asymptotic independence of the components, see Corollary 2, Remark 3-Remark 6, and Appendix B. For further work in the area relating branching processes and PCR consult [27], [28] and [25] while for statistical problems involving mutation rates see [9].

We begin with a brief description of the branching process. We denote by $\{Z_n : n \geq 0\}$ the Galton-Watson process initited by a single ancestor $Z_0 \equiv 1$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{\xi_{n,j}, j \geq 1, n \geq 1\}$ denote a double array of integer valued i.i.d. random variables with probability distribution $\{p_j : j \geq 0\}$, i.e.

$$P(\xi_{1,1} = k) = p_k. \tag{1.1}$$

The random variable $\xi_{n,j}$ is interpreted as the number of children produced by the j^{th} parent in the $(n-1)^{th}$generation. The branching process $\{Z_n : n \geq 1\}$ is iteratively defined as follows: for $n \geq 1$

$$Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{n,j}. \tag{1.2}$$

Let $m = E(Z_1)$. It is well known that if $m > 1$ (*i.e.* the process is supercritical), then $Z_n \to \infty$ with positive probability and that the probability that the process becomes extinct, namely q, is less than one. Furthermore, q satisfies the functional equation

$$f(q) = q, \tag{1.3}$$

where for $0 \le s \le 1$,

$$f(s) = \sum_{j \ge 0} s^j p_j. \tag{1.4}$$

Also, $q = 0$ if and only if $p_0 = 0$. We will assume in our paper that $1 < m < \infty$.

Let us denote by $\mathcal{F}_n$ the $\sigma-$field generated by the sequence $\{Z_0, Z_1, \cdots, Z_n\}$. Let $W_n = \frac{Z_n}{m^n}$. Then it is known that $\{(W_n, \mathcal{F}_n) : n \ge 0\}$ is a non-negative martingale sequence, and an important classical result due to Kesten and Stigum is that it converges to a non-degenerate limit W if and only if $E(Z_1 \log Z_1) < \infty$, see, for example, [2], Theorem 1, page 24. Furthermore, as can be seen from [2], Corollary 4, p.36, almost surely on the survival set S we have $0 < W < \infty$. If $E(Z_1 \log Z_1) = \infty$, then there exists a sequence of constants $\{c_n\}$ such that the normalized sequence $W_n^{SH} = \frac{Z_n}{c_n}$ has a non-degenerate limit. The constants c_n are called the Senata constants, and we denote the almost sure limit of W_n^{SH} by W^{SH}. Furthermore, Theorem 3, p. 30, and Corollary 1, p.52, of [2] imply that almost surely on the survival set, $0 < W^{SH} < \infty$. We have indexed the random variables in this last setting by SH, as Heyde showed the almost sure convergence in Seneta's earlier result, which provided only convergence in distribution.

The quantity $R_n = Z_{n-1}^{-1} Z_n$ is known as Nagaev's estimator of the mean m, and it has been shown in [16] to be the maximum likelihood estimator when only (Z_{n-1}, Z_n) are observed. The law of large numbers (consistency) and the central limit theorem (asymptotic normality) associated with R_n have previously been studied under the first and second moment, respectively. The law of the iterated logarithm associated with R_n was also known under the assumption of finite $(2+\delta)$ moments for the offspring distribution (see, for instance, [18] [19]). Now consider the vector $\mathbf{R}_{n,r(n)} \equiv (R_n, R_{n-1}, \cdots, R_{n-r(n)+1})$. When $r(n)$ is independent of n, then $\mathbf{R}_{n,r(n)}$ is a vector of fixed length, and the law of large numbers for the vector can be obtained by looking at the components individually. However, when $r(n) \nearrow \infty$ various difficulties emerge, but we obtain a number of strong law results for $\mathbf{R}_{n,r(n)}$, and hence also R_n, via continuity theorems applied to our functional limit theorems. Furthermore, we study the conditional and unconditional complete convergence of these functional law of large numbers.

The central limit theorem for the multigenerational process is somewhat surprising. For instance, if $r(n) = 2$, we show that the limit distribution of $(\sqrt{Z_{n-1}}(R_n - m), \sqrt{Z_{n-2}}(R_{n-1} - m))$ is bivariate normal with mean vector $\mathbf{0}$ and covariance matrix $\sigma^2 I_2$, where I_2 is the identity matrix of order 2; that is the components are asymptotically independent. The problem is more complex when $r(n) \nearrow \infty$, but the asymptotic independence of different components exists even in this setting. We also establish the functional version of such a result when $r(n) \nearrow \infty$ under no hypothesis other than the finiteness of the second moment.

The rate of convergence in the classical central limit theorem based on an independent and identically distributed (i.i.d.) sequence of centered random variables is given by the law of the iterated logarithm(LIL). In particular, the LIL studies the almost sure large values of the normalized partial sums, and the factor $(\log \log n)^{\frac{1}{2}}$ in the denominator is required to provide a finite and nonzero limsup. The functional version of the classical LIL is due to Strassen in [32], and represents a landmark in the study of limit theorems in probability. In the so-called other LIL, namely Chung's LIL ([5]) for centered partial sums, the emphasis is not on large values for the normalized partial sums, but rather on their small values and the rate of escape from zero. Chung's result also contains the factor $(\log\log n)^{\frac{1}{2}}$, but now it appears in the numerator and one needs small deviation probabilities at the functional level to do the analysis. The functional version of Chung's result appeared in the important paper [34].

The classical LIL has a long history, which we will not repeat here, as it is fairly well known, and Strassen's LIL is also a widely known result. However, since Chung's LIL and its functional version due to Wichura are perhaps less well known, we include a few remarks and references in this area. These are far from comprehensive, but are intended to motivate the statements of Theorems 5 and 6 in Section 2 which generalize Wichura's result to multiple generations of the branching process $\{Z_n : n \geq 0\}$.

We begin with Chung's law in the context of a sequence of *i.i.d.* random variables $\{X_j : j \geq 1\}$. Let $S_k = \sum_{j=1}^k X_j$ and set $M_n = \max_{1\leq k\leq n} |S_k|$ where $E(X_1) = 0,\ 0 < \sigma^2 = E(X_1^2) < \infty$. The distributional behavior of M_n was given by Erdos and Kac in [14], where they established that

$$\frac{M_n}{n^{\frac{1}{2}}\sigma} \xrightarrow{d} V, \tag{1.5}$$

with

$$P(V \leq x) = \frac{4}{\pi}\sum_{i\geq 0}\frac{(-1)^i}{2i+1}\exp(-\frac{(2i+1)\pi^2}{8x^2}). \tag{1.6}$$

Chung ([5]), under a finite third moment assumption, established a law of the iterated logarithm for the convergence in (1.5). More precisely, Chung proved that if $E|X_1|^3 < \infty$, then with probability one

$$\liminf_{n\to\infty}\sqrt{\frac{\log\log n}{n\sigma^2}}M_n = \frac{\pi}{\sqrt{8}}. \tag{1.7}$$

In his proof, Chung used the Erdos-Kac result, and it is no small coincidence that the constant $\frac{\pi}{\sqrt{8}}$ in (1.7) is the square root of the constant in the exponent of the $i = 0$ term in the series given in (1.6). This is the constant determining the asymptotics of $x^{-2}\log P(V \leq x)$ as $x \downarrow 0$, and is the small ball constant in this setting. It determines the constant in (1.7), because it is the cutoff point for the convergence or divergence in the Borel-Cantelli arguments used in this situation. It is also the so-called small ball constant for Wiener measure since the distribution in (1.6) is also the distribution of the norm of Brownian motion when time is restricted to $[0, 1]$.

Pakshirajan ([30]) established the above result under a $(2 + \delta)$ moment assumption, and in Chung's review of this paper he raised the question as to whether the result was true under a finite second moment assumption. Jain and Pruitt ([21]) answered Chung's question affirmatively, and somewhat later in [8], the analogue of (1.7) was established for i.i.d. finite dimensional random vectors with only two moments. Furthermore, the rate of escape constant was also shown to be the small ball constant of the related limiting Brownian motion. However, the main task in the paper [8] was not to establish these results alone, but rather the more refined idea to use these small deviation probability estimates to obtain an extension of Chung's LIL which provides a speed of convergence result refining Strassen's LIL. This is a very elegant result, and certainly deserves more discussion, but we mention it here only to further motivate the fact that small ball constant's and small deviation probabilities appear in a variety of ways. Of course, they appeared in Chung's original work as well, but there three moments were assumed for the proof, and it was unknown if the limiting constant in this setting existed with fewer moments, and also whether it was independent of the moment condition.

Wichura ([34]), still working under slightly more than two moments, established the functional form of (1.7) and also obtained the related result for the Brownian motion. This result has recently been generalized to a number of different settings. These include [6], which deals with such results for symmetric stable processes having stationary independent increments, [23] which studies the fractional Brownian motion case, and [24], where Wichura's FLIL is generalized to certain stochastic integrals. Again, a central feature in these results is that the small ball probability constant and the rate of escape constant are equal. Of course, in view of the invariance principle, the small deviation probabilities for Brownian motion can be obtained via a series expansion in (1.6),but for other processes far less is known about such probabilities. In fact, a challenging step to extending the Wichura FLIL to other processes, or classes of processes, is usually to determine the necessary small deviation probability estimates required. In this paper this is accomplished using known rates of convergence of the Prokhorov metric for the invariance principle. The use of Prokhorov's metric in connection with the LIL appeared earlier in a result in [22], and also works here to overcome the additional difficulties imposed when working with partial sums from successive generations of a branching process, rather than partial sums from a fixed i.i.d. sequence of random variables.

In the context of branching processes, set

$$M_{n,Z_{n-1}} = \{\max_{1\leq k\leq Z_{n-1}} |\sum_{j=1}^{k} \frac{(\xi_{n,j}-m)}{\sqrt{\sigma^2 k}}|\}, \tag{1.8}$$

where $\sigma^2 = var(Z_1)$. Let us also define $\mathbf{M}_n \equiv (M_{n,Z_{n-1}}, M_{n-1,Z_{n-2}}, \cdots M_{n-r(n)+1,Z_{n-r(n)}}, 0, 0, \cdots)$. Theorems 5 and 6 in Section 2 present the functional version of (1.7) for the sequence $\{\mathbf{M}_n\}$ and $\{M_{n,Z_{n-1}}\}$.

The $(\log\log n)^{\frac{1}{2}}$ factor in the classical LIL, and also Strassen's FLIL, is a consequence of two facts. Roughly speaking they are as follows. First, to look at almost sure limit theorems for the maximum of the partial sums, it is sufficient to look along geometric subsequences of the partial sums. This takes care of one of the logarithms, and the second is eliminated through various comparisions, which one can hope will allow one to exploit the large deviations of the Gaussian limit distribution in the CLT for these partial sums. A similar comment applies to Chung's LIL and Wichura's functional generalization, except now it is the exponential tail behavior of the small deviation probabilities for the sup-norm of Brownian motion given by (1.6) that eliminate the second logarithm. However, in the branching process setup we are working with a triangular array of random variables and hence there is no fixed sequence of random variables from which one can extract geometric subsequences. As a result, our analogue of Strassen's functional LIL and Wichura's functional LIL will only involve a factor of $(\log n)^{\frac{1}{2}}$ and not an iterated logarithm. Hence, we speak of Strassen's functional law of the logarithm and Wichura's functional law of the logarithm, even though we may some times refer to such laws as LIL's. Heyde ([19]) established an analogue of the classical LIL for the partial sums associated with R_n, and here we establish functional versions of Strassen [32] and Chung-Wichura [34] type for statistics built from vectors $\mathbf{R}_{n,r(n)}$ in R^∞. Finally, it is perhaps worth mentioning that one might think that the factor $(\log n)^{\frac{1}{2}}$ in our results is natural since the sample size in each generation is Z_n, and $\log\log Z_n$ behaves likes $\log n$ for supercritical processes when the process does not die out. However, this reasoning is incorrect, since similar results hold as stated for triangular arrays of centered i.i.d. random variables with third moments, as long as the n^{th} row has $n^{8+\delta}$ terms, i.e. the papers [29], [12], and [13] can be consulted for suitable rates of convergence of the Prokhorov metric in the invariance principle under a variety of moment conditions. In particular, (1.7) of [13] implies suitable rates for our purposes for uniformly bounded i.i.d. triangular arrays if the row lengths are shortened to $n^{2+\delta}$ terms.

Finally, one can view these results as a first step in obtaining detailed limit theorems for triangular arrays of correlated random variables when functionals are constructed from successive rows. A key technical tool that proves very useful to achieve this, in this context, concerns the harmonic moments of logarithms of generation sizes. This, along with Lemma 8, allows us to establish the "almost sure results" obtained here under fairly sharp moment conditions. We also are then able to study multiple generations simultaneously using an iterative approach which is also new as far as we are aware. The key is to get sharp estimates for functionals defined on the sucessive generations. The feasibility of extending these tools to other models used in such areas like clinical trials is under investigation.

The paper is organized as follows: Section 2 develops the basic notation and states the main results of the paper. Section 3 contains the proofs of the laws of large numbers. Section 4 is devoted to the proof of the functional central limit theorem, while Section 5 deals with the functional versions of Strassen's law of the logarithm. Section 6 is devoted to the proof of the Chung-Wichura law of the logarithm. Section 7 is an appendix, which contains a useful result on the harmonic moment of $(LZ_n)^r$ for $r > 0$, and Section 8 contains a few simulation results.

2 Notation, Assumptions, and Main Results

In this section, we state the main results of the paper. Our goal is to obtain functional limit theorems for supercritical branching processes based on $r(n)$-generations, where $1 \leq r(n) \leq n$. In particular, the integer sequence $\{r(n)\}$ may approach infinity as n goes to infinity, and these functional limit theorems will enable the study of the random vector $\mathbf{R}_{n,r(n)} \equiv (R_n, R_{n-1}, \ldots, R_{n-r(n)+1}, 0, 0, \ldots) \in R^\infty$ as $n \to \infty$. We begin by describing the processes in which we embed $\mathbf{R}_{n,r(n)}$. On the set $\{Z_{n-1} > 0\}$, and for $0 \leq t \leq 1$, we define

$$Y_{n,Z_{n-1}}(t) = \frac{1}{Z_{n-1}} \sum_{j=1}^{\lfloor tZ_{n-1} \rfloor} (\xi_{n,j} - m) + (tZ_{n-1} - \lfloor tZ_{n-1} \rfloor) \frac{1}{Z_{n-1}} (\xi_{n,\lfloor tZ_{n-1} \rfloor + 1} - m), \tag{2.1}$$

and on the set $\{Z_{n-1} = 0\}$ we define $Y_{n,Z_{n-1}}(t) = 0,\ 0 \leq t \leq 1$. We view each $Y_{n,Z_{n-1}}(\cdot)$ as an element of the set of all continuous functions on [0,1] that vanish at 0, which we denote by $C_0[0,1]$. Then $C_0[0,1]$ is a Banach space with the usual supremum norm, and we want to study the asymptotic behavior of the $r(n)$-dimensional random vector $(Y_{n,Z_{n-1}}(\cdot), Y_{n-1,Z_{n-2}}(\cdot), \ldots, Y_{n-r(n)+1,Z_{n-r(n)}})$ as $n \to \infty$. Since $r(n)$ may well converge to infinity, it is useful for these purposes to define

$$\mathbf{Y}_{n,r(n)}(\cdot) = (Y_{n,Z_{n-1}}(\cdot), Y_{n-1,Z_{n-2}}(\cdot), \ldots, Y_{n-r(n)+1,Z_{n-r(n)}}(\cdot), 0, 0, \ldots), \tag{2.2}$$

where the zeros in the previous vector are the zero function in $C_0[0,1]$. Hence $\mathbf{Y}_{n,r(n)}(\cdot)$ is an element of $(C_0[0,1])^\infty$, where

$$(C_0[0,1])^\infty \equiv \prod C_0[0,1] \tag{2.3}$$

is the infinite cartesian product of $C_0[0,1]$ with the product topology. Since the product topology is metrizable with the metric

$$d_\infty(\mathbf{x},\mathbf{y}) = \sum_{k\geq 1} \frac{1}{2^k}\frac{||x_k - y_k||}{1+||x_k-y_k||}, \tag{2.4}$$

where $||\cdot||$ is the supremum norm on $C_0[0,1]$, it is sufficient to study the convergence in the d_∞ metric. We now state our result concerning the functional law of large numbers. Let S denote the survival set of the process, and S^c its complement.

Theorem 1. *Assume that $E(Z_1) < \infty$ and that $1 \leq r(n) \leq n$. Then,*

$$\lim_{n\to\infty} d_\infty(\mathbf{Y}_{n,r(n)}, \mathbf{0}) = 0 \quad a.s. \ , \tag{2.5}$$

where $\mathbf{0} = (0,0,\cdots)$ and 0 is the constant function identically equal to 0.

Our next result concerns the strong law of large numbers where the sense of convergence is more demanding, and requires the uniform convergence of $\mathbf{Y}_{n,r(n)}$ to zero in $(C_0[0,1])^\infty$. More precisely, we define the non-negative function

$$\max \mathbf{Y}_{n,r(n)} \equiv \max_{1\leq j\leq r(n)} ||Y_{n-j+1,Z_{n-j}}||, \tag{2.6}$$

where as above $||\cdot||$ is the supremum norm on $C_0[0,1]$. Then we show the random quantity $\max \mathbf{Y}_{n,r(n)}$ converges completely to zero. That is, if

$$J(\epsilon) = \sum_{n\geq 1} P(\max \mathbf{Y}_{n,r(n)} > \epsilon), \tag{2.7}$$

and $J(\epsilon) < \infty$ for all $\epsilon > 0$, then we say $\max \mathbf{Y}_{n,r(n)}$ converges completely to zero. Of course, an easy application of the Borel-Cantelli then immediately implies we also have convergence of $\max \mathbf{Y}_{n,r(n)}$ to zero with probability one.

Theorem 2. *Let $1 \leq r(n) \leq n$, and assume $n - r(n) \geq (\log n)h(n)$, where $h(n) \to \infty$. Then the following hold:*

(a) If $E(Z_1^r) < \infty$ for some $r > 1$, then $\max \mathbf{Y}_{n,r(n)}$ converges completely to zero. In particular, with probability one

$$\lim_{n\to\infty} \max \mathbf{Y}_{n,r(n)} = 0. \tag{2.8}$$

(b) If $E(Z_1(LZ_1)^r) < \infty$ for some $r > 1$ and $\{r(n)\}$ also satisfies

$$\sum_{n\geq 1} r(n)(\log_e n)^r (n - r(n))^{-r} < \infty, \tag{2.9}$$

then $\max \mathbf{Y}_{n,r(n)}$ converges completely to zero and (2.8) holds with probabiity one.

Remark 1. *Let $\mathbf{m}_{n,r(n)}$ denote the vector in R^∞ whose first $r(n)$ entries are all m, with the rest being zero. Then the quantity $\mathbf{R}_{n,r(n)} = \mathbf{Y}_{n,r(n)}(1) + \mathbf{m}_{n,r(n)}$ has an interesting statistical interpretation. Indeed, for $1 \leq j \leq r(n)$, the j^{th} component of $\mathbf{Y}_{n,r(n)}(1) + \mathbf{m}_{n,r(n)}$ is the non-parametric maximum likelihood estimator (MLE) of the sample mean when the observation process is (Z_{n-j}, Z_{n-j+1}) [16]. Consider a statistical experiment in which generation sizes are observed at $r(n)$ generations by $r(n)$*

individuals working backwards from the n^{th} generation. Then each individual estimates the sample mean using the MLE, and hence the first $r(n)$ coordinates of $\mathbf{R}_{n,r(n)}$ represents these estimates. Our Theorems 1 and 2 help us understand the consistency property of these sample estimates. For example, if $\mathbf{m}$ denotes the vector in R^∞ all of whose entries are m, then on the survival set S Theorem 1 with $r(n) \to \infty$ implies the random vector $\mathbf{R}_{n,r(n)}$ is a consistent estimator of both $\mathbf{m}_{n,r(n)}$ and $\mathbf{m}$ in the product topology. This is immediate, since convergence in the product topology requires only that each coordinate converges. Furthermore, on the survival set S, Theorem 2 implies that $\mathbf{R}_{n,r(n)}$ is also a consistent estimator of both $\mathbf{m}_{n,r(n)}$ and $\mathbf{m}$, when we ask that consistency for $\mathbf{b} = (b_1, b_2, \ldots) \in R^\infty$ to mean

$$\max(\mathbf{R}_{n,r(n)} - \mathbf{b}) \equiv \max_{1 \le j \le r(n)} |Y_{n-j+1,Z_{n-j}}(1) + m - b_j| \tag{2.10}$$

converges to zero almost surely on S. Of course, (2.10) is most interesting when $r(n) \to \infty$, but if we change the definition of $\max(\mathbf{R}_{n,r(n)} - \mathbf{b})$ to be $\sup_{j \ge 1} |Y_{n-j+1,Z_{n-j}}(1) + m - b_j|$ for $\mathbf{b} = (b_1, b_2, \ldots) \in R^\infty$, where we assume $Y_{k,Z_{k-1}}(1) = 0$ for $k < 0$, then $\mathbf{R}_{n,r(n)}$ is no longer consistent for $\mathbf{m}$. Therefore Theorem 1 and Theorem 2 imply consistency results for the estimator $\mathbf{R}_{n,r(n)}$, and those from Theorem 1 involving the product topology one might call strong joint consistency, whereas those from Theorem 2 using

$$\max(\mathbf{R}_{n,r(n)} - \mathbf{b}) \equiv \max_{1 \le j \le r(n)} |Y_{n-j+1,Z_{n-j}}(1) + m - b_j| \tag{2.11}$$

would then be called uniform strong joint consistency. Of course, if one uses the above notions of consistency as given in Theorems 1 and 2 with $r(n) \to \infty$, then it is clear that uniform strong joint consistency always implies the strong joint consistency.

Our next corollary summarizes the consistency of the vector of mles. Its proof is immediate by setting $t = 1$

Corollary 1. *Under the conditions of Theorem 1 and 2 with $r(n) \to \infty$, the mles of the population means based on $r(n)$ successive generations satisfy strong joint consistency and uniform strong joint consistency on the survival set of the process.*

We now move on to study the functional central limit theorem (FCLT) associated with the sequence $\mathbf{R}_{n,r(n)}$. We first define the scaled version of the vector $\mathbf{Y}_{n,r(n)}$ and denote it by $\mathbf{X}_{n,r(n)}$. More precisely, let $\sigma^2 = Var(Z_1) < \infty$ denote the offspring variance. Let $0 \le t \le 1$, and on the set $\{Z_{n-1} > 0\}$ define

$$X_{n,Z_{n-1}}(t) = \frac{1}{\sigma\sqrt{Z_{n-1}}}\{\sum_{j=1}^{\lfloor tZ_{n-1} \rfloor} (\xi_{n,j} - m) + (tZ_{n-1} - \lfloor tZ_{n-1} \rfloor))(\xi_{n,\lfloor tZ_{n-1} \rfloor + 1} - m)\}, \tag{2.12}$$

and on the set $\{Z_{n-1} = 0\}$ define $X_{n,Z_{n-1}}(t) = 0$. Note that $X_{n,Z_{n-1}}(t) = \sqrt{\frac{Z_{n-1}}{\sigma^2}} Y_{n,Z_{n-1}}(t)$. Let

$$\mathbf{X}_{n,r(n)}(t) \equiv (X_{n,Z_{n-1}}(t), X_{n-1,Z_{n-2}}(t), \cdots X_{n-r(n)+1,Z_{n-r(n)}}(t), 0, 0, \cdots,). \tag{2.13}$$

We will use $\Rightarrow$ to denote the weak convergence. We are now ready to state the functional central limit theorem for the stochastic process $\mathbf{X}_{n,r(n)}(\cdot)$.

Theorem 3. *Assume that (i) $E(Z_1^2) < \infty$, (ii) $1 \leq r(n) \leq n$, and (iii) $\lim_{n\to\infty} r(n) = \infty$. Then, in the product topology on $(C_0[0,1])^\infty$, as $n \to \infty$*

$$\mathcal{L}(\mathbf{X}_{n,r(n)}|Z_{n-1} > 0) \Rightarrow \mathcal{L}(B_1, B_2, \cdots), \tag{2.14}$$

where the B_i's are independent standard Brownian motions.

Remark 2. *The fact that the limit law in our CLT is that given by an infinite product of i.i.d. Brownian motions implies that each coordinate, unless one decides to scale down the coordinates, has a limit law of equal significance. Hence without such scalings, the product topology is in some sense the natural topology for these theorems. More precisely, as in the case of the law of large numbers, by evaluating the functional $\mathbf{X}_{n,r(n)}(\cdot)$ at 1, we can deduce distributional results concerning the centered version of $\mathbf{R}_{n,r(n)}$ from those on $\mathbf{X}_{n,r(n)}$. In particular, we can deduce the joint asymptotic distribution of the centered MLEs. Since this result has consequences in inference for branching processes, we state this result as a corollary. Recall that,*

$$\sigma\mathbf{X}_{n,r(n)}(1) = (\sqrt{Z_{n-1}}(R_n - m), \sqrt{Z_{n-2}}(R_{n-1} - m), \cdots \sqrt{Z_{n-r(n)}}(R_{n-r(n)+1} - m), \cdots)I_{[Z_{n-1}>0]}. \tag{2.15}$$

Corollary 2. *Let $r(n) \equiv l$. Then under the condition that $E(Z_1^2) < \infty$, as $n \to \infty$,*

$$\mathcal{L}(\mathbf{X}_{n,r(n)})(1)|Z_{n-1} > 0) \Rightarrow (N_1, N_2, \cdots N_l, 0, 0, \cdots), \tag{2.16}$$

where N_i for $1 \leq i \leq l$, are independent normal random variables with mean 0 and variance 1. Of course, if $1 \leq r(n) \leq n$ and $\lim_{n\to\infty} r(n) = \infty$, then as $n \to \infty$ we have

$$\mathcal{L}(\mathbf{X}_{n,r(n)})(1)|Z_{n-1} > 0) \Rightarrow (N_1, N_2, \cdots N_l, N_{l+1}, \cdots), \tag{2.17}$$

where N_i for all $i \leq 1$, are independent normal random variables with mean 0 and variance 1.

Remark 3. *Returning to the PCR example mentioned in the introduction, in a real time PCR assay, data are available for every cycle during the entire experiment. One of the important questions experimentally is to statistically identify the end of the exponential phase. This corresponds to the change in the dynamics of the PCR process, namely from supercritical to critical. One approach to estimate the change point is to use data from k consecutive cycles to construct the confidence region for the $k-$dimensional vector $(m, m, \cdots m)$. Using Corollary 2, this confidence region can be approximated by the product of the one-dimensional confidence intervals. Then, the end of the exponential phase can be estimated by the cycle corresponding to the first time the confidence region includes the $k-$vector $(1, 1, \cdots, 1)$. The method of identifying the change point using $k > 1$ should be more reliable than $k = 1$ as explained below.*

Remark 4. *We performed simulations to evaluate the role of k, the number of confidence intervals that include 1, in correctly identifying the end of the exponential phase in a PCR experiment. To describe the simulation experiment, we first briefly describe the branching process model for PCR data. The branching process model for the PCR data has an offspring distribution with support on $\{1, 2\}$. Let $P(Z_1 = 2) = p$. Then, $m = 1 + p$. We consider two distinct cases. First, we study when p changes from positve to 0 at a particular generation. We call this a discontinuous change point. Second, we study when p decreases "smoothly" to 0. We call this the continuous change point. Since in both the cases, p changes to 0 either smoothly or discontinuously, we will write $p(n)$ for p. We note that the linear phase was not included in our simulations.*

Remark 5. **(Results from a simulation study)** *Let $n^\star$ be the generation of the change point, i.e.*

$$n^\star = \inf\{n : p_n = 0\}.$$

1. *Case 1. Here $n^\star = 11$.*
$$p_n = \begin{cases} .6, & 1 \le n \le 10 \\ 0, & n \ge 11 \end{cases}$$

2. *Case 2. Here $n^\star = 16$.*
$$p_n = \begin{cases} .4, & 1 \le n \le 15 \\ 0, & n \ge 16 \end{cases}$$

3. *Case 3. Here $n^\star = 25$.*
$$p_n = \begin{cases} .8, & 1 \le n \le 17 \\ .8 - .1(n-17), & 18 \le n \le 24 \\ 0, & n \ge 25 \end{cases}$$

We conducted 1000 simulations for each of the three cases described above. Results of the simulation study are included in the Appendix B. As the results show, it is clear that large values of k correspond to a more precise estimate of $n^\star$ and this happens more often for the case $k > 1$ than the case $k = 1$.

Remark 6. *In the case 3 described above, since the probability distribution changes across cycles, typically one cannot model the data during the entire exponential phase using a single homogeneous branching process. However, if the population size is large when p starts decreasing, it is difficult to statistically identify the differences between the non-homogeneous and the homogeneous branching processes.*

We now move on to describe our results concerning the laws of the logarithm. Let

$$K_1 = \{f \in C_0[0,1] : f(t) = \int_0^t g(s)ds, 0 \le t \le 1, \int_0^1 g^2(s)ds \le 1\}. \tag{2.18}$$

In view of the role played by K_1 in the study of the law of the iterated logarithm for i.i.d. random variables, K_1 is called the Strassen's set. Finally, for any $A \subset C_0[0,1]$ and $x \in C_0[0,1]$, let

$$d_1(x, A) = \inf_{y \in A} ||x - y||. \tag{2.19}$$

For any sequence $\{f_n\} \equiv \{f_n : n \ge 1\}$, let $C(\{f_n\})$ denote the set of all limit points of the sequence $\{f_n\}$. $C(\{f_n\})$ is called the cluster set of $\{f_n\}$. To state our result concerning the law of the logarithm, we need an infinite dimensional version of K_1 and throughout the paper we let $Lt = \max\{1, \log_e t\}$ for $t \ge 0$.

We now describe the limit set K_∞ in $(C_0[0,1])^\infty$ for the processes $\{X_{n,r(n)}\}$ properly normalized. That is,

$$K_\infty = \{(f_1, f_2, \ldots) \in (C_0[0,1])^\infty : \; f_k(t) = \int_0^t g_k(s)ds \text{ for } \mathrm{k} \ge 1, \text{and} \sum_{\mathrm{k} \ge 1} \int_0^1 \mathrm{g}_\mathrm{k}^2(\mathrm{s})\mathrm{ds} \le 1\}. \tag{2.20}$$

Our next result concerns an analogue of Strassen's law for these $(C_0[0,1])^\infty$ valued processes, provided the product topology is used on the range space.

Theorem 4. *Assume $E(Z_1^2(L(Z_1))^r) < \infty$ for some $r > 4$, that $1 \le r(n) \le n$, and we also have $\lim_{n\to\infty} r(n) = \infty$. Then*

$$P(\lim_{n\to\infty} d_\infty(\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}, K_\infty) = 0) = 1, \tag{2.21}$$

where the d_∞-distance from a point to a set is defined as usual. In addition, if S denotes the survival set of the process and clustering is determined with respect to the product topology, then we have

$$P(C(\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}\}) = K_\infty|S) = 1. \tag{2.22}$$

If we define for $\mathbf{f} = (f_1, f_2, \dots) \in (C_0[0,1])^\infty$ the map

$$\pi_l(\mathbf{f}) = (f_1, \cdots, f_l), \quad 1 \le l < \infty,$$

then setting $l = 1$, the previous theorem, and the continuity of the $\pi_l(\cdot)'s$ easily imply an analogue of Strassen's functional law for data based on the generations $n-1$ and n, Setting $t = 1$ in this result implies a result of Heyde under weaker conditions than is available in [18]. We state this as a corollary. Of course, an analogue holds for every $l, \quad 1 \le l < \infty$.

Corollary 3. *Assume that $E(Z_1^2(LZ_1)^r) < \infty$ for some $r > 4$. Then*

$$P(\lim_{n\to\infty} d_1(\frac{X_{n,Z_{n-1}}}{\sqrt{2Ln}}, K_1) = 0) = 1, \tag{2.23}$$

where for any set $A \subset C_0[0,1]$ and $f \in C_0[0,1]$, $d_1(f, A)$ is given by (2.18). Furthermore, if S denotes the survival set of the process and clustering is determined with respect to the uniform norm on $C_0[0,1]$, then we have

$$P(C(\{\frac{X_{n,Z_{n-1}}}{\sqrt{2Ln}}\}) = K_1|S) = 1. \tag{2.24}$$

Remark 7. *The fact that $r > 4$ in Theorem 4 and Corollary 3 results from the use of standard estimates for the Prokhorov distance in the classical invariance theorem. That these estimates are essentially best possible can be seen from [4] and also [31]. Thus an attempt at reducing $r > 4$ to , say $r > 1$, would seem to require a substantially different approach than what we use here. For rates of convergence, under stronger moment conditions, the reader should consult the papers [12, 13].*

We next describe the maximal processes used in connection to a generalization of a theorem of Wichura. To describe these results we need further notation. Let $\mathcal{M}$ denote the non-decreasing functions on [0,1] into $[0, \infty]$ such that $f(0) = 0$, and f is right continuous on (0,1). If $\{h_n\} \subseteq \mathcal{M}$, we say $\{h_n\}$ converges to $h \in \mathcal{M}$ if $\lim_n h_n(t) = h(t)$ for all $t \in [0,1]$ where $h(\cdot)$ is continuous from [0,1] into $[0, \infty]$. The limit set in Wichura's LIL is

$$\mathcal{K}_1 = \{h \in \mathcal{M} : \int_0^1 h^{-2}(s)ds \le 1\}. \tag{2.25}$$

Furthermore, it is easy to see from classical arguments, (see [15]), that the convergence in $\mathcal{M}$ mentioned above can be metrized through the use of the Lévy metric on the non-decreasing functions h^* on $(-\infty, \infty)$ which are right continuous on (0,1), $h^*(0) = 0$, $h^*(1) \le 1$, $h^*(t) = h^*(1)$ for $t \ge 1$, and such that $h^*(t) = h^*(0)$ for $t < 0$. That is, if $\lambda(s) = s/(1+s)$ for $0 \leq s \leq \infty$, then the metric ρ on $\mathcal{M}$, which is of interest, is given by

$$\rho(h, g) = d_L(h^*, g^*), \tag{2.26}$$

where

$$h^*(s) = \lambda(h(s)), \quad 0 \leq s \leq 1, \tag{2.27}$$

and d_L is Lévy's metric. Of course, for given $h \in \mathcal{M}$ the function h^* used in (2.26) is assumed to be such that $h^*(s) = 0$ for $s < 0$, $h^*(s) = h^*(1)$ for $s > 1$, and given by (2.27) on [0,1]. $(\mathcal{M}, \rho)$ is also separable since the subprobabilities on [0,1] are separable in Lévy's metric. We also define the maximal process related to $X_{n,Z_{n-1}}(\cdot)$ by

$$M_{n,Z_{n-1}}(t) = \sup_{0 \leq s \leq t} |X_{n,Z_{n-1}}(s)|, \quad 0 \leq t \leq 1. \tag{2.28}$$

We are, of course, interested in the infinite dimensional version of the maximal processes. To this end, we first define the vector maximal process $\mathbf{M}_{n,r(n)}$ analogous to (2.28) as follows:

$$\mathbf{M}_{n,r(n)}(t) = (M_{n,Z_{n-1}}(t), M_{n-1,Z_{n-2}}(t), \cdots, M_{n-r_n+1,Z_{n-r(n)}}(t), 0, 0, \cdots). \tag{2.29}$$

The infinite dimensional Chung-Wichura limit set is as follows:

$$\mathcal{K}_\infty = \{(h_1, h_2, \ldots,) \in \mathcal{M}^\infty : \sum_{k=1}^{\infty} \int_0^1 h_k^{-2}(s) ds \leq 1\}, \tag{2.30}$$

where $\mathcal{M}^\infty$ is the infinite cartesian product of $\mathcal{M}$. The topology on $\mathcal{M}^\infty$ is the product topology which is complete and separable in the topology given by the metric

$$\rho_\infty(\mathbf{f}, \mathbf{g}) = \sum_{k \geq 1} \frac{1}{2^k} \frac{\rho(f_k, g_k)}{1 + \rho(f_k, g_k)}, \tag{2.31}$$

where $\mathbf{f} = (f_1, f_2, \ldots)$, $\mathbf{g} = (g_1, g_2, \ldots)$ and ρ is the metric given in (2.25). Our next result presents the functional form of the Chung-Wichura law for samples drawn from the past r_n consecutive generations of the branching processes. In all that follows in connection with the Chung-Wichura results, we'll always assume $c_2 = \frac{\pi^2}{8}$.

Theorem 5. *Assume $E(Z_1^2(L(Z_1))^r) < \infty$ for some $r > 4$, that $1 \leq r(n) \leq n$, and we also have $\lim_{n\to\infty} r(n) = \infty$. Let S denote the survival set of the process. Then,*

$$P(\lim_{n\to\infty} \rho_\infty(\sqrt{\frac{Ln}{c_2}} \mathbf{M}_{n,r(n)}, \mathcal{K}_\infty) = 0|S) = 1. \tag{2.32}$$

Furthermore, when clustering is determined with respect to the ρ_∞-topology, then

$$P(C(\{\sqrt{\frac{Ln}{c_2}} \mathbf{M}_{n,r(n)}\}) = \mathcal{K}_\infty|S) = 1. \tag{2.33}$$

Our next result, which is an immediate corollary to Theorem 5, states the analogue of the Chung-Wichura law for the process based on data in generations $n-1$ and n. It follows using the analogue of the maps defined following the statement of Theorem 4, except now π_l take $\mathcal{M}^\infty$ continuously onto the l-fold product space $\mathcal{M}^l$. We state a result for $l = 1$, but there are obvious analogues for all l, $\;1 \leq l < \infty$.

Theorem 6. *Assume that $E(Z_1^2(LZ_1)^r) < \infty$ for some $r > 4$. Let S denote the survival set of the process. Then*

$$P(\lim_{n\to\infty} \rho(\sqrt{\frac{Ln}{c_2}} M_{n,Z_{n-1}}, \mathcal{K}_1) = 0|S) = 1. \tag{2.34}$$

Furthermore, when clustering is determined with respect to the ρ-topology, then

$$P(C(\{\sqrt{\frac{Ln}{c_2}} M_{n,Z_{n-1}}\}) = \mathcal{K}_1|S) = 1. \tag{2.35}$$

3 Functional Laws of Large Numbers

In this section we provide proofs of the functional strong laws of large numbers in Theorems 1 and 2. We start with the proof of Theorem 1, which we split into several lemmas.

Lemma 1. *Let $\epsilon > 0$. Then there exists $r_0 = r_0(\epsilon)$ such that for all $r \geq r_0(\epsilon)$ and all $n \geq 1$*

$$P(\max_{1\leq k\leq r} |\sum_{j=1}^{k} (\xi_{n,j} - m)| > 2r\epsilon) \leq 2P(|\sum_{j=1}^{r} (\xi_{n,j} - m)| > r\epsilon). \tag{3.1}$$

Proof. If $\epsilon > 0$ is given, then $E(\xi_{n,1}) < \infty$ and the weak law of large numbers implies there exists a $k_0(\epsilon)$ such that for all $k \geq k_0(\epsilon)$ and $n \geq 1$ we have

$$P(|\sum_{j=1}^{k} (\xi_{n,j} - m)| > k\epsilon) < \frac{1}{2}. \tag{3.2}$$

Thus for $r \geq k_0(\epsilon)$ we have for all $n \geq 1$ that

$$\max_{1\leq k\leq r} P(|\sum_{j=1}^{k} (\xi_{n,j} - m)| \geq r\epsilon) \leq \max(1/2, \max_{1\leq k\leq k_0(\epsilon)} P(|\sum_{j=1}^{k} (\xi_{n,j} - m)| \geq r\epsilon)). \tag{3.3}$$

Now taking $r_0(\epsilon) \geq k_0(\epsilon)$ sufficiently large, we have for all $n \geq 1$ that

$$\max_{1\leq k\leq r} P(|\sum_{j=1}^{k} (\xi_{n,j} - m)| \geq r\epsilon) \leq 1/2. \tag{3.4}$$

Thus by Ottaviani's inequality, ([7]), we have (3.1), and the lemma is proven. ∎

Our next lemma establishes the almost sure convergence of $Y_n(1)$, and is a straight forward consequence of the Senata-Heyde result ([2], page 30).

Lemma 2. $\lim_{n\to\infty} Y_{n,Z_{n-1}}(1) = 0$ *a.s.*

Proof: Let S denote the survival set of the process and S^c its complement. Then on S^c, by definition, there exists an $n_0(\omega)$ such that $Z_{n-1} = 0$ for all $n \geq n_0(\omega)$. Hence, by definition of the process, $Y_{n,Z_{n-1}}(1) = 0$ on S^c. We now deal with the set S. Note that on S, $Z_n > 0$ for all $n \geq 1$ almost surely. Hence,

$$\frac{Z_n}{Z_{n-1}} = (\frac{Z_n c_n^{-1}}{Z_{n-1} c_{n-1}^{-1}}) \frac{c_n}{c_{n-1}} \tag{3.5}$$

Now, by Theorem 3 of [2]

$$\lim_{n\to\infty}\left(\frac{Z_n c_n^{-1}}{Z_{n-1}c_{n-1}^{-1}}\right) = \frac{W(\omega)}{W(\omega)} = 1, \tag{3.6}$$

where the last equality follows from $W(\omega) > 0$ a.s. on S. Also, again from [2],

$$\lim_{n\to\infty} \frac{c_n}{c_{n-1}} = m. \tag{3.7}$$

Hence, $Y_{n,Z_{n-1}}(1) \to 0$ *a.s.* on S, completing the proof of the lemma. ∎

An interesting consequence of Lemma 2 is the following estimate of Z_n on a set S_0 (defined below) of probability $1-q$. Define,

$$S_0 = \{\omega : \lim_{n\to\infty} \frac{Z_n(\omega)}{Z_{n-1}(\omega)} = m\}. \tag{3.8}$$

Then, from the proof of Lemma 2, it follows that $P(S\Delta S_0) = 0$ and $S^c \cap S_0 = \phi$. Also, on S_0 the following hold: for every $1 < \beta < m$, and all $\omega \in S_0$, there is a $n_0(\omega)$ such that for all $n \geq n_0(\omega) + 1$

$$Z_n(\omega) > \beta Z_{n-1}(\omega) > Z_{n-1}(\omega) \tag{3.9}$$

and

$$Z_n(\omega) \geq \max\{Z_0(\omega), \cdots, Z_{n-1}(\omega)\}. \tag{3.10}$$

Thus (3.9) and (3.10) imply that for all $\omega \in S_0$ and $n \geq n_0(\omega)$,

$$Z_n(\omega) \geq \beta^{n-n_0} Z_{n_0}(\omega), \tag{3.11}$$

where $n_0 = n_0(\omega)$.

Let us denote by $\mathcal{F}_0$ the trivial $\sigma-$field, i.e. $\mathcal{F}_0 = \{\phi, \Omega\}$, and for $n \geq 1$ let

$$\mathcal{F}_n = \sigma(\{\xi_{k,j} : j \geq 1\} : 1 \leq k \leq n). \tag{3.12}$$

Furthermore, let

$$B_n(\epsilon) = \{\sup_{1\leq k \leq Z_{n-1}} |\sum_{j=1}^{k}(\xi_{n,j} - m)| > 2Z_{n-1}\epsilon\}. \tag{3.13}$$

Our next lemma is one way to express the role of the branching property of $\{Z_n : n \geq 0\}$, and allows us to complete the proof of Theorem 1. It also is used in the study of complete convergence in Theorem 2.

Lemma 3. *Let $\epsilon > 0$ and let $\{\xi, \xi_n : n \geq 1\}$ be an i.i.d. sequence defined on the probability space $(\Omega_1, \mathcal{G}, Q)$, which is different from $(\Omega, \mathcal{F}, P)$, the probability space supporting $\{Z_n : n \geq 0\}$. Furthermore, assume that $\mathcal{L}(\xi) = \mathcal{L}(Z_1)$. Then there exists a finite random variable n_0 on $(\Omega, \mathcal{F}, P)$ such that for all $n \geq n_0(\omega)$ we have with P-probability one that*

$$\begin{aligned} P(B_n(\epsilon)|\mathcal{F}_{n-1})(\omega) &\leq 2Q(|\sum_{j=1}^{Z_{n-1}(\omega)}(\xi_j - m))| \geq \epsilon Z_{n-1}(\omega)) && (3.14)\\ &= 2P(|\sum_{j=1}^{Z_{n-1}}(\xi_j - m)| \geq \epsilon Z_{n-1}|Z_{n-1})(\omega). && (3.15)\end{aligned}$$

Of course, all terms in (3.14-15) are understood to be zero if $Z_{n-1}(\omega) = 0$.

Proof. By the Markov property of $\{Z_n : n \geq 0\}$, it follows that

$$P(B_n(\epsilon)|\mathcal{F}_{n-1}) = P(B_n(\epsilon)|Z_{n-1}). \tag{3.16}$$

If $\omega \in S^c$, then eventually $Z_{n-1}(\omega) = 0$ and by definition of the $Y_{n,Z_{n-1}}$ process, both the LHS and RHS of (3.14) are then zero. Hence for $\omega \in S^c$ we set $n_0(\omega) = \min\{k \geq 1 : Z_k(\omega) = 0\}$. Thus, it is remains to establish the validity of the lemma on S_0 since $P(S\Delta S_0) = 0$. To this end, let $\omega \in S_0$ and $r_0 = r_0(\epsilon)$ be as in Lemma 1. Then there exists an $n_0 = n_0(\omega)$ such that for all $n \geq n_0$, $Z_{n-1}(\omega) > r_0(\epsilon)$. Hence first by (3.16) and the branching property, and then by Lemma 1 and the branching property, we have

$$\begin{aligned} P(B_n(\epsilon)|\mathcal{F}_{n-1})(\omega) &= Q(\sup_{1\leq k\leq Z_{n-1}(\omega)} |\sum_{j=1}^{k}(\xi_j - m)| > 2Z_{n-1}(\omega)\epsilon) && (3.17)\\ &\leq 2Q(|\sum_{j=1}^{Z_{n-1}(\omega)}(\xi_j - m)| > \epsilon Z_{n-1}(\omega)) && (3.18)\\ &= 2P(|\sum_{j=1}^{Z_{n-1}}(\xi_j - m)| > \epsilon Z_{n-1}|Z_{n-1})(\omega), && (3.19)\end{aligned}$$

yielding the lemma. ∎

Proof of Theorem 1. First we will show that with probability one

$$\lim_{n\to\infty} ||Y_{n,Z_{n-1}}|| = 0. \tag{3.20}$$

If $\omega \in S^c$ then $Z_{n-1} = 0$ for some $n \geq n_0$ and hence on S^c we have

$$\lim_{n\to\infty} ||Y_{n,Z_{n-1}}|| = 0. \tag{3.21}$$

If $\omega \in S$, then by the conditional Borel-Cantelli Lemma, it is sufficient to show that

$$\sum_{n\geq 1} P(||Y_{n,Z_{n-1}}|| > \epsilon|\mathcal{F}_{n-1}) < \infty. \tag{3.22}$$

Using Lemma 3 and that $B_n(\epsilon) = \{||Y_{n,Z_{n-1}}|| > \epsilon\}$, Lemma 1 implies it is sufficient to show that

$$\sum_{n\geq 1} P(|\sum_{j=1}^{Z_{n-1}}(\xi_j - m) > \epsilon Z_{n-1}|\mathcal{F}_{n-1}) < \infty \quad a.s. \tag{3.23}$$

on S. Now, by Lemma 2

$$P(\{\omega : \{|\frac{Z_n}{Z_{n-1}} - m| > \epsilon\}\ i.o\} \cap S) = 0, \tag{3.24}$$

which by the conditional Borel-Cantelli Lemma is equivalent to the finiteness of the LHS of (3.23) a.s. on S. Thus we have established (3.20), and the theorem follows as the convergence in the metric in (2.4) requires only coordinatewise convergence for finitely many coordinates with probability one. ∎

Proof of Theorem 2. The proof will again proceed with several lemmas, some of which will also be of use for later proofs. The first lemma is essentially Theorem 3 of [AN, p.41].

Lemma 4. *Let $\{Z_n : n \geq 0\}$ be a supercritical Galton-Watson process with $Z_0 = 1$. Then there exists a constant $\gamma \in (0,1)$ such that*

$$\lim_{n\to\infty} P(Z_n = k)/\gamma^n = \nu_k, \tag{3.25}$$

where $0 \leq \nu_k < \infty$ for all $k \geq 1$.

Proof. If $p_1 = P(Z_1 = 1) \neq 0$, then $f'(q) = p_1$, and since the process is supercritical we have $f'(q) \in (0,1)$. Thus with $\gamma = f'(q)$, Theorem 3.1 of [AN, p. 41] implies the lemma. When $p_1 = 0$ there are two cases to consider, namely $p_0 = 0$ and $p_0 \neq 0$. If $p_0 = 0$, then $Z_n \geq 2^n$ and hence eventually for any fixed k we have $P(Z_n = k) = 0$. Thus the lemma also holds in this case with γ any number in $(0,1)$ and $\nu_k = 0$.

Hence there remains the case $p_0 \neq 0,\ p_1 = 0$. In this last case, since the process is supercritical, we have $0 < q < 1$, and again we have $0 < f'(q) < 1$, so the result in [AN] cited above implies the lemma. ∎

Our next lemma provides estimates which are useful in connection with the complete convergence in Theorem 2. We use the notation of Lemma 3, where $\{\xi, \xi_j : j \geq 1\}$ are i.i.d random variables on the probability space $(\Omega_1, \mathcal{G}, Q)$.

Lemma 5. *Let $T_k = \sum_{j=1}^{k}(\xi_j - m)$. Then there exists a $k_0(\epsilon, \mathcal{L}(\xi))$, where $\mathcal{L}(\xi)$ is the law of ξ, such that for all $k \geq k_0$*

$$Q(|T_k| > k\frac{\epsilon}{2}) \leq 16\epsilon^{-2}k^{-1}E((\xi - m)^2 I(|\xi - m| \leq k)) + kQ(|\xi - m| \geq k). \tag{3.26}$$

Proof. Let

$$\hat{T}_k = \sum_{j=1}^{k}(\xi_j - m)I(|\xi_j - m| \leq k) \ \text{ for all } k \geq 1. \tag{3.27}$$

Now observe that

$$Q(|T_k| > \frac{k\epsilon}{2}) \leq Q(|\hat{T}_k| > \frac{k\epsilon}{2}) + Q(|T_k - \hat{T}_k| > 0). \tag{3.28}$$

Now,

$$Q(|T_k - \hat{T}_k| > 0) \leq kQ(|\xi - m| \geq k), \tag{3.29}$$

and it remains to estimate $Q(|\hat{T}_k| > \frac{k\epsilon}{2})$. Observe that, since $E(\xi - m) = 0$, there exists a $k_0(\epsilon)$ such that $k \geq k_0(\epsilon)$ implies $|E(\xi - m)I(|\xi - m| \leq k))| \leq \frac{\epsilon}{4}$. Hence,

$$\begin{aligned} Q(|\hat{T}_k| > \frac{k\epsilon}{2}) &\leq Q(|\hat{T}_k - E(\hat{T}_k)| > \frac{k\epsilon}{4}) & (3.30)\\ &\leq 16\epsilon^{-2}k^{-1}E((\xi - m)^2 I(|\xi - m| \leq k)). & (3.31) \end{aligned}$$

Thus (3.26) holds for all $k \geq k_0$. ∎

To finish the proof of Theorem 2, we first note that

$$\begin{aligned} P(\sup_{1\leq j\leq r(n)} ||Y_{n-j+1,Z_{n-j}}|| > \epsilon) &\leq \sum_{j=1}^{r(n)} P(||Y_{n-j+1,Z_{n-j}}|| > \epsilon) & (3.32)\\ &= \sum_{j=1}^{r(n)} P(||Y_{n-j+1,Z_{n-j}}|| > \epsilon \quad Z_{n-j} > 0) & (3.33)\\ &= I_n + II_n, & (3.34) \end{aligned}$$

where

$$I_n = \sum_{j=1}^{r(n)} \sum_{k=1}^{n_0(\epsilon)} P(||Y_{n-j+1,Z_{n-j}}|| > \epsilon \ \ Z_{n-j} = k) \tag{3.35}$$

$$\leq \sum_{j=1}^{r(n)} \sum_{k=1}^{n_0(\epsilon)} P(Z_{n-j} = k), \tag{3.36}$$

and

$$II_n = \sum_{j=1}^{r(n)} \sum_{k \geq n_0(\epsilon)+1} P(||Y_{n-j+1,Z_{n-j}}|| > \epsilon \ \ Z_{n-j} = k). \tag{3.37}$$

We will first study I_n. By Lemma 4 above, there exists $0 < M < \infty$, $0 \leq \nu_k < \infty$, and $0 < \gamma < 1$ such that

$$P(Z_{n-j} = k) \leq M(\nu_k + 1)\gamma^{n-j} \tag{3.38}$$

for $1 \leq k \leq n_0(\epsilon)$ and all $n > j$. Note that M may need to be large, but the set of $k's$ where the inequality holds is finite, and hence $M < \infty$ is possible. Since $r(n) \geq (\log n)(h(n)$ where $\lim_{n\to\infty} h(n) = \infty$ in both parts of the theorem, we have easily have

$$\sum_{n\geq 1} r(n)\gamma^{n-r(n)} < \infty, \tag{3.39}$$

and hence for all $n_0(\epsilon) < \infty$ we have

$$\sum_{n\geq 1} I_n \leq M \sum_{n\geq 1} \sum_{k=1}^{n_0(\epsilon)} (\nu_k + 1) r(n)\gamma^{n-r(n)} < \infty. \tag{3.40}$$

Thus it remains to deal with II_n. The first case we consider is when the assumptions in part-a hold. Since the conclusions of part-a do not involve r and we have $r > 1$, it suffices to assume $1 < r \leq 2$ when we write the proof. Thus by a result of von Bahr and Esseen in [33] we have

$$E(|\sum_{j=1}^{k} (\xi_j - m)/k|^r) \leq B_r E(|\xi - m)|^r) k^{-(r-1)}, \tag{3.41}$$

for an absolute constant B_r. Now we also have

$$P(||Y_{n-j+1,Z_{n-j}}|| > \epsilon \ \ Z_{n-j} = k) = Q(\max_{1\leq l\leq k} |\sum_{j=1}^{l} (\xi_j - m)| > k\epsilon) P(Z_{n-j} = k), \tag{3.42}$$

and applying Lemma 1 and Markov's inequality (3.41) and (3.42) combine to imply $\sum_{n\geq 1} II_n < \infty$ provided

$$\sum_{n\geq r(n)+1} \sum_{j=1}^{r(n)} \sum_{k\geq n_0(\epsilon)+1} k^{-(r-1)} P(Z_{n-j} = k) \leq \sum_{n\geq r(n)+1} \sum_{j=1}^{r(n)} E(Z_{n-j}^{-(r-1)} I(Z_{n-j} > 0)) < \infty. \tag{3.43}$$

Since $r - 1 \leq 1$ and the process is supercritical, the proof of (4) in Theorem 2 of [19] implies

$$E(Z_{n-j}^{-(r-1)} I(Z_{n-j} > 0)) \leq (\gamma_{HB}^{n-j})^{r-1}, \tag{3.44}$$

where $0 < \gamma_{HB} < 1$. Combining (3.43-44) we thus have $\sum_{n\geq 1} II_n < \infty$ since

$$\sum_{n\geq 1}\sum_{j=1}^{r(n)}(\gamma_{HB}^{n-j})^{r-1} \leq \sum_{n\geq 1} r(n)(\gamma_{HB}^{n-r(n)})^{r-1} < \infty \tag{3.45}$$

when $r > 1, 0 < \gamma_{HB} < 1, 1 \leq r(n) \leq n$ and $n - r(n) \geq (\log n)h(n)$ with $\lim_{n\to\infty} h(n) = \infty$. Thus part-a is proven.

Turning to part-b and using the notation of Lemma 3 we see that

$$P(||Y_{n-j+1,Z_{n-j}}|| > \epsilon \quad Z_{n-j} = k) = Q(\max_{1\leq l\leq k} |\sum_{j=1}^{l}(\xi_j - m)| > k\epsilon)P(Z_{n-j} = k). \tag{3.46}$$

Now taking $n_0(\epsilon) \geq \max(r_0(\frac{\epsilon}{2}), k_0(\epsilon))$, where $r_0(\cdot)$ is defined as in Lemma 1, and $k_0(\epsilon)$ is as in Lemma 5, we have for $k \geq n_0(\epsilon) + 1$,

$$P(||Y_{n-j+1,Z_{n-j}}|| > \epsilon \quad Z_{n-j} = k) \leq 2Q(|\sum_{j=1}^{k}(\xi_j - m)| > k\epsilon)P(Z_{n-j} = k). \tag{3.47}$$

Thus, $\sum_{n\geq 1} II_n < \infty$ if

$$\sum_{n\geq r(n)+1}\sum_{j=1}^{r(n)-1}\sum_{k\geq n_0(\epsilon)+1}(E(k^{-1}(\xi - m)^2 I(|\xi - m| \leq k)) + kQ(|\xi - m| \geq k))P(Z_{n-j} = k) < \infty. \tag{3.48}$$

Now, by Markov's inequality,

$$kQ(|\xi - m| \geq k) \leq \frac{E(\phi(|\xi - m|))}{(Lk)^r}, \tag{3.49}$$

where $\phi(t) = t(L(t))^r$ for $t \geq 0$. We now deal with the other term inside the sum in (3.48). Let $c_0 \geq e^e$ be sufficiently large that

$$E(k^{-1}(\xi - m)^2 I(|\xi - m| \leq k)) \leq II_n(A) + II_n(B) + II_n(C), \tag{3.50}$$

where

$$II_n(A) = E(k^{-1}(\xi - m)^2 I(|\xi - m| \leq c_0)) \leq c_0^2 k^{-1}, \tag{3.51}$$

$$II_n(B) = E(k^{-1}(\xi - m)^2 I(c_0 \leq |\xi - m| \leq k(Lk)^{-r})) \leq E(|\xi - m|)(Lk)^{-r}, \tag{3.52}$$

and

$$\begin{aligned} II_n(C) &= E(k^{-1}(\xi - m)^2 I(c_0 \leq k(Lk)^{-r} \leq |\xi - m| \leq k)) & (3.53)\\ &\leq E(|\xi - m|(L(|\xi - m|)^r)(L(k(Lk)^{-r}))^{-r} I_{(k\geq c_0))}. & (3.54)\end{aligned}$$

The last inequality holds if $c_0 > e^e$ and $t \geq c_0$ is sufficiently large so that we have $Lt - rLLt \geq \frac{1}{2}Lt$. Combining (3.48),(3.49), and the estimates in (3.50-54) we see that

$$\sum_{n\geq r(n)+1} II_n \leq C \sum_{n\geq r(n)+1}\sum_{j=0}^{r(n)-1} E(LZ_{n-j}^{-r}(Z_{n-j} > 0)), \tag{3.55}$$

where C is a finite positive constant. Now using the harmonic moment results for LZ_n from Appendix A, we see that

$$\sum_{n\geq 1} II_n \leq C \sum_{n\geq 1} \frac{r(n)(\log_e n)^r}{(n-r(n))^r} < \infty \tag{3.56}$$

where C is a possibly different finite positive constant and the last series converges by assumption. This completes the proof of Theorem 2. ∎

Using our Theorems 1 and 2 one can study limit theorems for the uncentered version of $Y_{n,Z_{n-1}}$, scaled by constants c_{n-1} instead of Z_{n-1}. In particular, if $1 < E(Z_1) = m < \infty$ the Senata constants mentioned in section one can be used for c_n, and if we also have $E(Z_1 LZ_1) < \infty$, then we can take $c_n = m^n$. To this end, we define the appropriately modified processes

$$Y^m_{n,c_{n-1}}(t) = \frac{1}{c_{n-1}} \sum_{j=1}^{\lfloor tZ_{n-1}\rfloor} \xi_{n,j} + (tZ_{n-1} - \lfloor tZ_{n-1}\rfloor)\frac{1}{c_{n-1}}(\xi_{\lfloor tZ_{n-1}\rfloor+1}). \tag{3.57}$$

Then, under the asssumption that $1 < m < \infty$, for every fixed t, as $n \to \infty$, $Y^m_{n,c_{n-1}}(t) \to mtV$ a.s., where V is a non-degenerate random variable. Indeed, $V = W$ if $E(Z_1 LZ_1) < \infty$ and $V = W^{SH}$ if $E(Z_1 LZ_1) = \infty$. Of course, the limit is non-zero only on the survival set S. Furthermore, one also then has that $||Y_{n,c_{n-1}}||$ converges almost surely to mV as $n \to \infty$. Our next result, a corollary of Theorem 1, shows that in the product topology the stochastic process

$$\mathbf{Y}^m_n = (Y^m_{n,c_{n-1}}(t), Y^m_{n-1,c_{n-2}}(t), \cdots, Y^m_{n-r(n)+1,c_{n-r(n)}}(t), 0, 0, \cdots) \tag{3.58}$$

converges to an appropriate limit almost surely.

Proposition 1. *Assume that $E(Z_1) < \infty$ and $1 \leq r(n) \leq n$ with $\lim_{n\to\infty} r(n) = \infty$. Let*

$$\mathbf{L} = (mtV, mtV, \cdots,). \tag{3.59}$$

where $V = W$ if $E(Z_1 LZ_1) < \infty$ and $V = W^{SH}$ if $E(Z_1 LZ_1) = \infty$. Then,

$$\lim_{n\to\infty} d_\infty(\mathbf{Y}^m_n, \mathbf{L}) = 0 \quad a.s.. \tag{3.60}$$

Furthermore, the limit $\mathbf{L}$ is strictly positive and finite almost surely only on the non-extinction set S and is zero almost surely on S^c.

Remark 8. *The previous proposition is a functional generalization of a classical limit theorem for a supercritical Galton-Watson process.*

Proof. Let

$$N_n(t) = \sum_{j=1}^{\lfloor tZ_{n-1}\rfloor} \xi_{n,j} + (tZ_{n-1} - \lfloor tZ_{n-1}\rfloor)(\xi_{\lfloor tZ_{n-1}\rfloor+1}), \quad 0 \leq t \leq 1. \tag{3.61}$$

Then with the constants c_n defined as above we have

$$Y^m_{n,c_{n-1}}(t) = N_n(t)/c_{n-1}, \quad 0 \leq t \leq 1, \tag{3.62}$$

and

$$Y_{n,Z_{n-1}}(t) = N_n(t)/Z_{n-1} - mt, \quad 0 \leq t \leq 1. \tag{3.63}$$

Since we are asking for almost sure convergence in the product topology in Proposition 1, it is easy to see that it suffices to show that

$$\lim_{n\to\infty} \sup_{0\le t\le 1} |Y^m_{n,c_{n-1}}(t) - mtV| = 0 \tag{3.64}$$

with probability one. Now this limit is immediate almost surely on the complement of the survival set S, since $Y^m_{n,c_{n-1}}(\cdot)$, for all large n, and V both equal zero almost surely there. On the set S we have $Z_{n-1} > 0$ and

$$\begin{aligned}
\sup_{0\le t\le 1} |Y^m_{n,c_{n-1}}(t) - mtV| &= \sup_{0\le t\le 1} |\frac{N_n(t)}{c_{n-1}} - mtV| && (3.65)\\
&\le \sup_{0\le t\le 1} [|\frac{N_n(t)}{c_{n-1}} - mt\frac{Z_{n-1}}{c_{n-1}}| + mt|\frac{Z_{n-1}}{c_{n-1}} - V|] && (3.66)\\
&\le \frac{Z_{n-1}}{c_{n-1}} \sup_{0\le t\le 1} |\frac{N_n(t)}{Z_{n-1}} - mt| + m|\frac{Z_{n-1}}{c_{n-1}} - V| && (3.67)\\
&= \frac{Z_{n-1}}{c_{n-1}} ||Y_{n,Z_{n-1}}|| + m|\frac{Z_{n-1}}{c_{n-1}} - V|. && (3.68)
\end{aligned}$$

Letting $n \to \infty$ in the above, and applying Theorem 1 and the Seneta-Heyde and Kesten-Stigum results as explained in the introduction, the proposition is proved. ∎

4 Functional Central Limit Theorem

In this section we provide a proof of the functional central limit theorem. The proof is based on a lemma for weak convergence in infinite product spaces. It seems such a lemma should be in the literature, but we could not find it. Hence we include it for completeness and its independent interest. We begin with a bit of notation. Let (S, d) be a complete separable metric space and μ be a Borel probability measure on (S, d) and $\pi : S \to S$ be Borel measurable. Define,

$$\mu^\pi(A) = \mu(\pi^{-1}(A)) \tag{4.1}$$

for all Borel sets A of (S, d). Let S^∞ denote the infinite product space with a typical point $\mathbf{s} = (s_1, s_2, \cdots)$. The product topology on S^∞ is metrizable with metric

$$d_\infty(\mathbf{s}, \mathbf{t}) = \sum_{j\ge 1} \frac{1}{2^j} \frac{d(s_j, t_j)}{1 + d(s_j, t_j)}, \tag{4.2}$$

where $\mathbf{s}, \mathbf{t} \in S^\infty$. If $\mathbf{q} = (q_1, q_2, \cdots,)$ is a point in S, we define the mapping $\pi_l : S^\infty \to S^\infty$, for $l \ge 1$, by

$$\pi_l(\mathbf{s}) = (s_1, s_2, \cdots, s_l, q_{l+1}, q_{l+2}, \cdots,). \tag{4.3}$$

We now state a lemma concerning weak convergence in product spaces. A proof is provided for the sake of completeness.

Lemma 6. *Let $\{\mu_n : n \ge 1\}$ and μ_∞ be Borel probability measures on (S^∞, d_∞). Then $\{\mu_n : n \ge 1\}$ converges weakly to μ_∞ if and only if $\mu_n^{\pi_l}$ converges weakly to $\mu_\infty^{\pi_l}$ for all $l \ge 1$.*

Proof. To prove weak convergence, we need to verify that (see, for example, [11])

$$\lim_{n\to\infty} \int_{S^\infty} f d\mu_n = \int_{S^\infty} f d\mu_\infty \tag{4.4}$$

for $f : S^\infty \to S^\infty$ satisfying $||f|_{BL} \le 1$, where

$$||f||_{BL} = \sup_{\mathbf{s}\in S^\infty} |f(\mathbf{s})| + \sup_{\mathbf{s}\neq\mathbf{t}\in S^\infty} \frac{|f(\mathbf{s}) - f(\mathbf{t})|}{d(\mathbf{s},\mathbf{t})}. \tag{4.5}$$

First note that the mappings $\{\pi_l : l \ge 1\}$ are continuous and hence Borel measurable. Then for $l \ge 1$ we have,

$$\int_{S^\infty} f(\mathbf{s}) d\mu_n - \int_{S^\infty} f(\mathbf{s}) d\mu_\infty = I_1 + I_2 + I_3, \tag{4.6}$$

where

$$I_1 = \int_{S^\infty} (f(\mathbf{s}) - f(\pi_l(\mathbf{s})) d\mu_n, \tag{4.7}$$

$$I_2 = \int_{S^\infty} f(\pi_l(\mathbf{s})) d(\mu_n - \mu_\infty), \tag{4.8}$$

and

$$I_3 = \int_{S^\infty} (f(\pi_l(\mathbf{s})) - f(\mathbf{s})) d\mu_\infty. \tag{4.9}$$

Let $\epsilon > 0$ and $l_0(\epsilon) > 0$ be such that $\sum_{j\ge l_0(\epsilon)} 2^{-j} \le \frac{\epsilon}{2}$. This implies that for $l \ge l_0(\epsilon)$,

$$d(\mathbf{s}, \pi_l(\mathbf{s})) = \sum_{j\ge n_0(\epsilon)+1} \frac{1}{2^j} \frac{d(s_j, q_j)}{1 + d(s_j, q_j)} \le \frac{\epsilon}{2}; \tag{4.10}$$

thus, for $||f||_{BL} \le 1$, we have

$$\limsup | \int_{S^\infty} f d(\mu_n - \mu_\infty)| \le \frac{\epsilon}{2} + \limsup_{n\to\infty} | \int_{S^\infty} f(\pi_l(\mathbf{s}) d(\mu_n - \mu_\infty)| + \frac{\epsilon}{2}. \tag{4.11}$$

Since $\epsilon > 0$ is arbitrary, the asserted weak convergence follows. Finally, if $\mu_n \Rightarrow \mu_\infty$, then since $\{\pi_l : l \ge 1\}$ are continuous, by the continuous mapping theorem $\mu_n^{\pi_l} \Rightarrow \mu_\infty^{\pi_l}$. ∎

Proof of Theorem 3. Let μ denote Wiener measure on $C_0[0,1]$ and μ_∞ be the infinite product measure formed by μ on $(C_0[0,1])^\infty$. Also let μ_n denote the law of $\mathbf{X}_{n,r(n)}$ when Z_{n-1} is conditioned to be stricty positive,i.e. for A a Borel subset of $(C_0[0,1[)^\infty$ we have

$$\mu_n(A) = P(\mathbf{X}_{n,r(n)} \in A | Z_{n-1} > 0).$$

By Lemma 6 with $S = C_0[0,1]$ and $\mathbf{q}$ the zero vector in $(C_0[0,1])^\infty$, it is sufficient to establish, for each $l \ge 1$, the weak convergence of $\mu_n^{\pi_l}$ to $\mu_\infty^{\pi_l}$. If we identify the range space of π_l with $(C_0[0,1])^l$ in the obvious way, then it suffices to show that on $(C_0[0,1])^l$ we have that

$$\lambda_n = \mathcal{L}(X_{n,Z_{n-1}}, X_{n-1,Z_{n-2}}, \cdots, X_{n-l+1,Z_{n-l}} | Z_{n-1} > 0)$$

converges weakly to $(\mu)^l$, the l-fold product of μ on that space.

To establish weak convergence of λ_n to $(\mu)^l$, it is sufficient, by Theorem 3.1 of Billingsley ([3]), to show for arbitrary continuity sets E_i of the Wiener measure on $C_0[0,1]$ that

$$\lim_{n\to\infty} \lambda_n(E_1 \times E_2 \times \cdots \times E_l) = \prod_{j=1}^{l} \mu(E_j). \tag{4.12}$$

We will now verify (4.12). To this end, set

$$\theta_n = (P(Z_{n-1} > 0))\lambda_n(\prod_{j=1}^{l} E_j). \tag{4.13}$$

Then,

$$\theta_n = E(\prod_{j=1}^{l} I_{A_{n,j}}), \tag{4.14}$$

where

$$A_{n,j} = \{X_{n-j+1, Z_{n-j}} \in E_j,\ Z_{n-j} > 0\} \tag{4.15}$$

for $1 \le j \le l$. Let $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_n = \sigma(\{\xi_{k,j} : j \ge 1\} : 1 \le k \le n)$ for $n \ge 1$. Also, to simplify the notation, write $A_{n,j} = A_j$, for $1 \le j \le l$. Now,

$$\begin{aligned} \theta_n &= E(E(\prod_{j=1}^{l} I_{A_j} | \mathcal{F}_{n-1})) & (4.16)\\ &= E(E(I_{A_1}|\mathcal{F}_{n-1}) \prod_{j=2}^{l} I_{A_j}). & (4.17) \end{aligned}$$

Now, setting $\beta_n \equiv E(I_{A_1}|\mathcal{F}_{n-1})$, we have

$$\beta_n = \Delta_n + \mu(E_1), \tag{4.18}$$

where $\Delta_n = E(I_{A_1}|\mathcal{F}_{n-1}) - \mu(E_1)$. Thus,

$$\theta_n = \mu(E_1)E(\prod_{j=2}^{l}(I_{A_j}) + e_n, \tag{4.19}$$

where

$$e_n = E(\prod_{j=2}^{l} I_{A_j}\Delta_n). \tag{4.20}$$

We will now show that $\lim_{n\to\infty} e_n = 0$. To this end, let $\epsilon > 0$ and note that

$$\begin{aligned} \limsup_{n\to\infty} e_n &\le \limsup_{n\to\infty} E(|\Delta_n| I_{Z_{n-2}>0}) & (4.21)\\ &\le I + II, & (4.22) \end{aligned}$$

where

$$I = \limsup_{n\to\infty} E(|P(A_1|\mathcal{F}_{n-1}) - \mu(E_1)| I_{(Z_{n-1}>0,\ Z_{n-2}>0)}), \tag{4.23}$$

and

$$II = \limsup_{n\to\infty} E(|P(A_1|\mathcal{F}_{n-1}) - \mu(E_1)|I_{(Z_{n-1}=0,\ Z_{n-2}=0)}). \tag{4.24}$$

Now observe that,

$$P(A_1|Z_{n-1}=j) = P(X_{n,Z_{n-1}} \in E_1|Z_{n-1}=j) = P(X_{n,j} \in E_1). \tag{4.25}$$

Hence, given $\epsilon > 0$, and that E_1 is a μ-continuity set, Donsker's invariance principle implies

$$|P(X_{n,j} \in E_1) - \mu(E_1)| \le \epsilon \tag{4.26}$$

for $j \ge j_0(\epsilon, E_1)$ independent of n, since $\{\xi_{n,j}, j \ge 1\}$ are i.i.d. with $E(\xi_{1,1}) = 0$ and $E(\xi_{1,1}^2) < \infty$. Now, using the Markov property of Z_n, we have

$$\begin{aligned}
I &= \limsup_{n\to\infty} E(|P(A_1|\mathcal{F}_{n-1}) - \mu(E_1)|I_{(Z_{n-1}>0\ Z_{n-2}>0)}) && (4.27)\\
&\le \limsup_{n\to\infty} E(|P(A_1|Z_{n-1}) - \mu(E_1)|I_{(Z_{n-1}>0\ Z_{n-2}>0)}) && (4.28)\\
&\le \lim_{n\to\infty} (\Sigma_{1,n} + \Sigma_{2,n}), && (4.29)
\end{aligned}$$

where

$$\Sigma_{1,n} = \sum_{j_1 \le j_0} \sum_{j_2 \ge 1} |P(A_1|Z_{n-1}=j_1) - \mu(E_1)| P(Z_{n-1}=j_1, Z_{n-2}=j_2), \tag{4.30}$$

$$\Sigma_{2,n} = \sum_{j_1 \ge j_0+1} \sum_{j_2 \ge 1} |P(A_1|Z_{n-1}=j_1) - \mu(E_1)| P(Z_{n-1}=j_1, Z_{n-2}=j_2). \tag{4.31}$$

Thus,

$$I \le \limsup_{n\to\infty} [2P(1 \le Z_{n-1} \le j_0) + \epsilon] = \epsilon, \tag{4.32}$$

where we used Lemma 4 to show that $\lim_{n\to\infty} P(1 \le Z_{n-1} \le j_0) = 0$. As for II, observe that

$$\begin{aligned}
II &\le 2 \limsup_{n\to\infty} \sum_{j\ge 1} P(Z_{n-2}=j, Z_{n-1}=0) && (4.33)\\
&\le \limsup_{n\to\infty} \sum_{j\ge 1} p_0^j P(Z_{n-2}=j) && (4.34)\\
&\equiv \limsup_{n\to\infty} [f_{n-2}(p_0) - f_{n-2}(0)] && (4.35)\\
&= 0, && (4.36)
\end{aligned}$$

since $f_{n-2}(p_0)$ and $f_{n-2}(0)$ both converge to q. Since $e_n \ge 0$, this implies $\lim_{n\to\infty} e_n = 0$.

Now returning to (4.19) and iterating we get,

$$\theta_n \equiv \prod_{j=1}^{l-1} \mu(E_j) E(I_{A_l}) + \sum_{j=0}^{l-2} \mu(E_j) e_{n-j}, \tag{4.37}$$

where $E_0 = C_0[0,1]$ and $e_{n-j} = E(\prod_{k=2+j}^{l} I_{A_j} \Delta_{n-j})$. Now, using an argument similar to the one used to prove $e_n \to 0$, one can show that $\lim_{n\to\infty} e_{n-j} = 0$.

Finally, note that

$$E(I_{A_l}) = P(X_{n-l+1,Z_{n-l}} \in E_l, Z_{n-l} > 0) \tag{4.38}$$
$$= \sum_{j\geq 1} P(X_{n-l+1,j}) \in E_l)P(Z_{n-l} = j) \tag{4.39}$$
$$= I_n + II_n + III_n, \tag{4.40}$$

where

$$I_n = \sum_{j=1}^{j_0(\epsilon,E_l)} P(X_{n-l+1,j} \in E_l)P(Z_{n-l} = j). \tag{4.41}$$

$$II_n = \sum_{j>j_0(\epsilon,E_l)} (P(X_{n-l+1,j} \in E_l) - \mu(E_l))P(Z_{n-l} = j), \tag{4.42}$$

and

$$III_n = \mu(E_l)P(Z_{n-l} > j_0(\epsilon, E_l)); \tag{4.43}$$

and $j_0(\epsilon, E_l)$ is such that for all $j \geq j_0(\epsilon, E_l)$

$$|P(X_{n-l+1,j} \in E_l) - \mu(E_l)| < \epsilon. \tag{4.44}$$

Existence of such a $j_0(\epsilon, E_l)$ follows by Donsker's invariance principle since $\{\xi_{n,j} : j \geq 1\}$ are all i.i.d. with $E(\xi_{n,1} = 0$ and $E(\xi^2_{n,1}) < \infty$. Using Lemma 4 we have $\lim_{n\to\infty} P(1 \leq Z_{n-l} \leq j_0(\epsilon, E_l)) = 0$, and hence it follows that $I_n \to 0$ as $n \to \infty$. Furthermore, using (4.44) and that $P(Z_{n-l} > j_0(\epsilon, E_l)) \to (1-q)$ it follows that $\limsup_{n\to\infty} II_n \leq \epsilon(1-q)$. Finally, $III_n \to \mu(E_l)(1-q)$. Thus, $\lim_{n\to\infty} E(I_{A_l}) = \mu(E_l)(1-q)$, which implies $\lim_{n\to\infty} \theta_n = (1-q)\prod_{j=1}^{l} \mu(E_j)$, and hence from (4.13) we have $\lim_{n\to\infty} \lambda_n = \prod_{j=1}^{l} \mu(E_j)$. Thus the theorem is proven. ■

5 Strassen's Functional Laws of the Logarithm

In this section we establish Strassen's law of the logarithm for supercritical Galton-Watson processes in the multiple generation setting. We begin with a general outline of the method of proof. As a first step, we will show with probability one that $\{\mathbf{X}_{n,r(n)}/(2Ln)^{1/2}\}$ is relatively compact with respect to the product topology on $(C_0[0,1])^\infty$. The next step will be to show that if $\mathbf{f} \notin K_\infty$, then with probability one $\mathbf{f}$ is not in the cluster set $C(\{\mathbf{X}_{n,r(n)}/(2Ln)^{1/2}\})$. Finally, conditioning on the suvival set, we show with probability one that every point of K_∞ is in the cluster set $C(\{\mathbf{X}_{n,r(n)}/(2Ln)^{1/2}\})$. Throughout, the topology on the range space $(C_0[0,1])^\infty$ is the product topology, which is separable and metric.

We begin with some notation that we will use extensively. For any subset A of a metric space M with metric d, A^δ is defined to be the set

$$A^\delta = \{y \in M | d(x,y) < \delta \text{ for some x} \in \text{A}\}. \tag{5.1}$$

In order not to cause confusion with this notation, we will henceforth usually write the complement of a typical set A as A', rather than A^c, as used earlier in connection with the survival set S. However, allowing abuse of this principle, we will continue to write the complement of S as before.

Our first lemma yields compactness of the cluster set K_∞.

Lemma 7. *K_∞ is a compact subset of $((C_0[0,1])^\infty, d_\infty)$.*

Proof. Let K^∞ denote the countably infinite product set $\prod K_1$, where in each coordinate

$$K_1 = \{f \in C_0[0,1] : f \in AC[0,1], ||f||_\mu^2 \le 1\}. \tag{5.2}$$

Here we write $AC[0,1]$ to denote the absolutely continuous functions in $C_0[0,1]$, and set $||f||_\mu^2 = \int_0^1 (f'(s))^2 ds$ for $f \in AC[0,1]$ such that f' is square integrable on $[0,1]$. Furthermore, we define $||f||_\mu^2 = \infty$ elsewhere on $C_0[0,1]$. Then it is well known that K_1 is a compact subset of $(C_0[0,1], ||\cdot||)$, see, for example, Strassen [32] or Lemma 2.1 of Kuelbs ([22]), and hence Tychnoff's Theorem immediately implies K^∞ is compact in $((C_0[0,1])^\infty, d_\infty)$. Since $K_\infty \subseteq K^\infty$, it is therefore enough to show that K_∞ is closed in $((C_0[0,1])^\infty, d_\infty)$.

Let $\{\mathbf{f}_n\} \in K_\infty \subseteq (C_0[0,1])^\infty$ and assume that

$$\lim_{n\to\infty} d_\infty(\mathbf{f}_n, \mathbf{f}) = 0 \tag{5.3}$$

for some $\mathbf{f} \in (C_0[0,1])^\infty$. Assume $\mathbf{f}_n = (f_{n,1}, f_{n,2}, \cdots)$ and $\mathbf{f} = (f_1, f_2, \cdots)$. Then for each $N \ge 1$, we have

$$\lim_{n\to\infty} \sup_{1\le j\le N} ||f_{n,j} - f_j|| = 0, \tag{5.4}$$

and the sequence $\{(f_{n,1}, f_{n,2}, \cdots f_{n,N}) : n \ge 1\}$ is contained in

$$K_N = \{(h_1, \cdots h_N) : \sum_{j=1}^N ||h_j||_\mu^2 \le 1, h_j \in AC[0,1], 1 \le j \le N\}. \tag{5.5}$$

Now K_N is compact in $(C_0[0,1])^N$ for all $N \ge 1$ since it is the limit set for Strassen's LIL for standard Brownian Motion on R^N. Thus $(f_1, \cdots f_N) \in K_N$ for all $N \ge 1$, which implies $\sum_{j\ge1} ||f_j||_\mu^2 \le 1$ and also that $f_j \in AC[0,1]$ for all $j \ge 1$ by definition of K_N and that K_N is closed in $(C_0[0,1])^N$. Thus, $f \in K_\infty$ and hence K_∞ is closed and compact. This completes the proof of the lemma. ∎

The next lemma is useful in our calculations several times.

Lemma 8. *Suppose $\phi(t) = t^2(Lt)^r$ where $r > 0$, $t \ge 0$, and as before $Lt = \max\{1, \log_e t\}$. If $E(\phi(Z_1)) < \infty$, $m = E(Z_1)$, and $\mathcal{L}(\xi) = \mathcal{L}(Z_1)$, where ξ is independent of Z_{n-1}, then there exists a finite positive constant $c(\xi, r)$, depending only on $r > 0$ and the law $\mathcal{L}(\xi) = \mathcal{L}(Z_1)$, such that*

$$Z_{n-1} P(|\xi - m| \ge Z_{n-1}^{1/2} | Z_{n-1}) I(Z_{n-1} > 0) \le c(r,\xi)/(LZ_{n-1})^r, \tag{5.6}$$

and

$$Z_{n-1} E(|\eta/Z_{n-1}^{1/2}|^3 | Z_{n-1}) I(Z_{n-1} > 0) \le c(r,\xi)/(LZ_{n-1})^r, \tag{5.7}$$

where

$$\eta = (\xi - m) I(|\xi - m| \le Z_{n-1}^{1/2}) - \mu_{n,Z_{n-1}}, \tag{5.8}$$

and

$$\mu_{n,Z_{n-1}} = E((\xi - m) I(|\xi - m| \le Z_{n-1}^{1/2}) | Z_{n-1}). \tag{5.9}$$

Proof. Since the terms to be dominated in (5.6) and (5.7) are all zero when $Z_{n-1} = 0$, and $Lt \geq 1$ for all $t \geq 0$, the result holds in this situation. Hence it suffices to prove the result when we assume $Z_{n-1} > 0$.

To simplify notation, let $\rho = \xi - m$. Then, since $\phi(t)$ is increasing for $t \geq 0$, we have by the conditional Markov inequality that

$$
\begin{aligned}
Z_{n-1}P(|\rho| \geq Z_{n-1}^{1/2}|Z_{n-1})I(Z_{n-1} > 0) &\leq Z_{n-1}E(|\rho|I(|\rho| \geq Z_{n-1}^{1/2})|Z_{n-1})/\phi(Z_{n-1}^{1/2}) && (5.10)\\
&= \int_{Z_{n-1}^{1/2}}^{\infty} t^2(Lt)^r dF_{|\rho|}(t)/(LZ_{n-1}^{1/2})^r && (5.11)\\
&\leq E(\phi(|\rho|))/(LZ_{n-1}^{1/2})^r && (5.12)\\
&\leq 2^r E(\phi(|\rho|))/(LZ_{n-1})^r, && (5.13)
\end{aligned}
$$

where in the last inequality we have used that $(Lt^{1/2})^r \geq (Lt)^r/2^r$ for $t \geq 0$ and $r > 0$. Thus (5.6) holds with $c(r, \xi) \geq 2^r E(\phi(|\xi - m|))$.

To verify (5.7) observe that

$$
\begin{aligned}
Z_{n-1}E(|\eta/Z_{n-1}^{1/2}|^3|Z_{n-1})I(Z_{n-1} > 0) &\leq Z_{n-1}^{-1/2}E(|\rho I(|\rho| \leq Z_{n-1}^{1/2}) - \mu_{n,Z_{n-1}}|^3|Z_{n-1}) && (5.14)\\
&\leq Z_{n-1}^{-1/2}\{a_{1,n} + a_{2,n}\}, && (5.15)
\end{aligned}
$$

where

$$a_{1,n} = E(|\rho I(|\rho| \leq Z_{n-1}^{1/2}) - \mu_{n,Z_{n-1}}|^2|\rho|I(|\rho| \leq Z_{n-1}^{1/2})|Z_{n-1}),$$

and

$$a_{2,n} = E(|\rho I(|\rho| \leq Z_{n-1}^{1/2}) - \mu_{n,Z_{n-1}}|^2|\mu_{n,Z_{n-1}}||Z_{n-1}).$$

Recalling $\mu_{n,Z_{n-1}}$ is $\sigma(Z_{n-1})$ measurable, we have

$$a_{1,n} \leq 2E(|\xi - m|^3 I(|\xi - m| \leq Z_{n-1}^{1/2})|Z_{n-1}) + 2\mu_{n,Z_{n-1}}^2 E(|\xi - m|),$$

and we also easily see that

$$a_{2,n} \leq |\mu_{n,Z_{n-1}}|E((\xi - m)^2).$$

Thus

$$Z_{n-1}E(|\eta/Z_{n-1}^{1/2}|^3|Z_{n-1})I(Z_{n-1} > 0) \leq Z_{n-1}^{-1/2}\{a_{3,n} + a_{4,n} + E((\xi - m)^2)|\mu_{n,Z_{n-1}}|\},$$

where $a_{3,n} = 2E(|\xi - m|^3 I(|\xi - m| \leq Z_{n-1}^{1/2})|Z_{n-1})$ and $a_{4,n} = 2\mu_{n,Z_{n-1}}^2 E(|\xi - m|)$. Since $|\mu_{n,Z_{n-1}}| \leq E(|\xi - m|)$ we see that

$$Z_{n-1}^{-1/2}\{a_{2,n} + a_{4,n}\} \leq c(r, \xi)/(LZ_{n-1})^r, \tag{5.16}$$

where $c(r, \xi)$ is a finite positve constant depending only on $r > 0$ and $\mathcal{L}(\xi)$.

Hence (5.7) will hold, and the lemma will be proved, if we show

$$Z_{n-1}^{-1/2}a_{3,n} = 2Z_{n-1}^{-1/2}E(|\xi - m|^3 I(|\xi - m| \leq Z_{n-1}^{1/2})|Z_{n-1}) \leq c(r, \xi)/(LZ_{n-1})^r, \tag{5.17}$$

where again $c(r, \xi)$ is a finite positve constant depending only on $r > 0$ and $\mathcal{L}(\xi)$. To verify (5.17) take $c_0 = c_0(r)$ such that $c_0 \geq e^e$ and if $t \geq c_0$, then $\log_e t - 2r\log_e(\log_e t) > (\log_e t)/2$. If $c_0 > Z_{n-1}$, then

$$Z_{n-1}^{-1/2}a_{3,n} \leq 2c_0 E(|\xi - m|^2)/Z_{n-1}^{1/2},$$

and again (5.7) will hold for a sufficiently large constant $c(r,\xi)$. Hence it remains to consider the case where $c_0 \le Z_{n-1}$. Thus we observe that

$$Z_{n-1}^{-1/2} a_{3,n} \le 2(A_{1,n} + A_{2,n}),$$

where

$$\begin{aligned} A_{1,n} &= E(|\xi - m|^2 |\xi - m| Z_{n-1}^{-1/2} I(0 < |\xi - m| \le Z_{n-1}^{1/2}/(LZ_{n-1})^r)|Z_{n-1}) & (5.18)\\ &\le E(|\xi - m|^2)/(LZ_{n-1})^r, & (5.19)\end{aligned}$$

and

$$\begin{aligned} A_{2,n} &= E(|\xi - m|^2 |\xi - m| Z_{n-1}^{-1/2} I(Z_{n-1}^{1/2}/(LZ_{n-1})^r) \le |\xi - m| \le Z_{n-1}^{1/2})|Z_{n-1}) & (5.20)\\ &\le E(\phi(|\xi - m|))/\{L(Z_{n-1}^{1/2}/(LZ_{n-1})^r)\}^r. & (5.21)\end{aligned}$$

Since $c_0 \le Z_{n-1}$, our choice of c_0 now allows us to complete the proof. ∎

Our next lemma gives some elementary properties of the set K_1 in $C_0[0,1]$ with respect to the sup-norm $||\cdot||$, which will be useful later in proving our version of Strassen's theorem.

Lemma 9. *If $\lambda \ge 1$ and $0 < \beta \le \delta/2$, then*

$$((\lambda (K_1)^\delta)')^\beta = (\lambda (K_1^\delta)')^\beta \subseteq \lambda (K_1^{\delta/2})'.$$

Proof. The set equality is obvious since multipling by $\lambda \ge 1$ is a one-to-one mapping. Hence we turn to the set inclusion. Now if $x \in (\lambda (K_1^\delta)')^\beta$, then there exists $y \in (K_1^\delta)'$ such that $||\lambda y - x|| < \beta$. We want x to be in $\lambda (K_1^{\delta/2})'$, so assume it is in the complement, i.e. $x \in \lambda (K_1^{\delta/2})$ since $\lambda \ne 0$. Hence if $x \in \lambda(K_1^{\delta/2})$ then $x = \lambda z$ where $z \in K_1^{\delta/2}$, so there exists $k \in K_1$ with $||z - k|| < \delta/2$. Thus from the above we have

$$||y - k|| \le ||y - x/\lambda|| + ||x/\lambda - k|| \le \beta/\lambda + ||z - k|| < \delta,$$

provided $\beta < \delta/2$ and $\lambda \ge 1$. This is a contradiction since $y \in (K_1^\delta)'$ and $k \in K$ impies $||y - k|| \ge \delta$. Thus the lemma holds. ∎

Before we state our next lemma, we recall from (2.13) that

$$\mathbf{X}_{n,r(n)} = (X_{n,Z_{n-1}}, X_{n-1,Z_{n-2}}, \cdots, X_{n-r(n)+1,Z_{n-r(n)}}, 0, 0, \cdots), \tag{5.22}$$

and for later use we define

$$\mathbf{X}_{n,l} = (X_{n,Z_{n-1}}, X_{n-1,Z_{n-2}}, \cdots, X_{n-l+1,Z_{n-l}}, 0, 0, \ldots). \tag{5.23}$$

Our next lemma concerns the relative compactness of the sequence $\{\mathbf{X}_{n,r(n)}/(2Ln)^{1/2}\}$ in the metric space $((C_0[0,1])^\infty, d_\infty)$.

Lemma 10. $P(\{\mathbf{X}_{n,r(n)}/(2Ln)^{1/2}\}$ *is relatively compact in* $(C_0[0,1])^\infty) = 1$.

Proof. Since $((C_0[0,1])^\infty, d_\infty)$ is separable, and K^∞ is a compact subset, it is sufficient to establish that for each $s > 0$ we have

$$\sum_{n \ge 1} P(\{\mathbf{X}_{n,r(n)}/(2Ln)^{1/2}\} \notin (K^\infty)^s) < \infty. \tag{5.24}$$

Let $s > 0$ be arbitrary but fixed. Let $\alpha > 0$ be such that $2\alpha = s/2(l+1)$, where $\sum_{k\geq l} 2^{-k} < s/2$. Then

$$P(\mathbf{X}_{n,r(n)}/(2Ln)^{1/2} \notin (K^{\infty})^s) \leq P(\mathbf{X}_{n,l}/(2Ln)^{1/2} \notin \prod(K_1)^{2\alpha}) \tag{5.25}$$

$$\leq \sum_{j=0}^{l-1} P(X_{n-j,Z_{n-j-1}}/(2Ln)^{1/2} \notin (K_1)^{2\alpha}). \tag{5.26}$$

Thus, to establish (5.24), it is sufficient to establish

$$\sum_{n\geq 1} P(\frac{X_{n,Z_{n-1}}}{(2Ln)^{\frac{1}{2}}} \notin (K_1)^{2\alpha}) < \infty. \tag{5.27}$$

Now, since $X_{n,Z_{n-1}} = 0$ when $Z_{n-1} = 0$, it follows that

$$P(\frac{X_{n,Z_{n-1}}}{(2Ln)^{\frac{1}{2}}} \notin (K_1)^{2\alpha}) = P(\frac{X_{n,Z_{n-1}}}{(2Ln)^{\frac{1}{2}}} \notin (K_1)^{2\alpha}, Z_{n-1} > 0) \tag{5.28}$$

Define

$$A_{n,\alpha} = \{\frac{X_{n,Z_{n-1}}}{(2Ln)^{\frac{1}{2}}} \notin (K_1)^{2\alpha}\}, \tag{5.29}$$

and with $c(r,\xi)$ as in Lemma 8 and $c_E = c_5(3,1)$ from Corollary 2 of [12] we also define

$$B_{n,\alpha} = \{c_E[c(r,\xi)/(LZ_{n-1})^r]^{1/4} < \frac{\alpha}{4}, Z_{n-1} \geq a_0\}, \tag{5.30}$$

$a \geq a_0 \geq e$ implies that

$$\sigma_a^2 = \int_{-a}^{a} t^2 dF_{(\xi-m)}(t) - (\int_{-a}^{a} t dF_{(\xi-m)}(t))^2 \tag{5.31}$$

$$\geq \sigma^2/4. \tag{5.32}$$

When $a = \sqrt{Z_{n-1}}$, we will abuse the notation and denote $\sigma^2_{\sqrt{Z_{n-1}}}$ by σ_n^2. Observe that, $\sigma_a^2 \leq \sigma^2$, and hence for $\sqrt{Z_{n-1}} \geq a_0$, we have $1 \leq \frac{\sigma}{\sigma_n} \leq 2$. Thus

$$P(A_{n,\alpha} \cap \{Z_{n-1} > 0\}) = P(A_{n,\alpha} \cap B_{n,\alpha}) + P(A_{n,\alpha} \cap \{Z_{n-1} > 0\} \cap B'_{n,\alpha}). \tag{5.33}$$

Now,

$$P(A_{n,\alpha} \cap B'_{n,\alpha}) \leq P(B'_{n,\alpha}) \tag{5.34}$$

$$\leq P(0 < Z_{n-1} \leq \max(a_0, \lfloor\Lambda\rfloor), \tag{5.35}$$

where $\lfloor\Lambda\rfloor = 1 + \exp\{16c_E^4 c(r,\xi)\alpha^{-4}\}$. Now, since Lemma 4 implies

$$\sum_{n\geq 1} P(Z_n \leq J) < \infty \tag{5.36}$$

for every $J < \infty$, it follows that

$$\sum_{n\geq 1} P(A_{n,\alpha} \cap \{Z_{n-1} > 0\} \cap B'_{n,\alpha}) < \infty. \tag{5.37}$$

Thus to complete the proof, we need to establish

$$\sum_{n\geq 1} P(A_{n,\alpha}\cap B_{n,\alpha}) < \infty. \tag{5.38}$$

To this end, define the truncated version of the $X_{n,Z_{n-1}}$ process as follows. If $Z_{n-1}>0$, define for $t=\frac{k}{Z_{n-1}}$and $1\leq k\leq Z_{n-1}$,

$$T_n(t) = (\sigma^2 Z_{n-1})^{-1/2}\sum_{j=1}^{k}((\xi_{n,j}-m)I(|\xi_{n,j}-m|\leq \sqrt{Z_{n-1}})-\mu_{n,Z_{n-1}}); \tag{5.39}$$

the function is linearly interpolated for other values of t with $T_n(0)=0$; furthermore,

$$\mu_{n,Z_{n-1}} = E((\xi-m)I(|(\xi-m)|\leq Z_{n-1}^{\frac{1}{2}}|Z_{n-1})); \tag{5.40}$$

if $Z_{n-1}=0$ then set $T_n(t)=0$ for all $t\in[0,1]$. Now, returning to (5.38)

$$\begin{aligned} P(A_{n,\alpha}\cap B_{n,\alpha}) &= P(\{\frac{X_{n,Z_{n-1}}}{(2Ln)^{\frac{1}{2}}}\notin K^{2\alpha}\}\cap B_{n,\alpha}) && (5.41)\\ &\leq I_n + II_n, && (5.42)\end{aligned}$$

where

$$I_n = P(\{\frac{T_n}{(2Ln)^{\frac{1}{2}}}\notin K^{\alpha}\}\cap B_{n,\alpha}), \tag{5.43}$$

and

$$II_n = P(\{||X_{n,Z_{n-1}}-T_n||\geq \alpha((2Ln)^{\frac{1}{2}})\}\cap B_{n,\alpha}). \tag{5.44}$$

We will first deal with II_n. Since $\alpha(2Ln)^{\frac{1}{2}}>0$, we have

$$\begin{aligned} II_n &\leq P(\{||X_{n,Z_{n-1}}-T_n||\geq \alpha((2Ln)^{\frac{1}{2}})\}\cap B_{n,\alpha}) && (5.45)\\ &\leq P(\sup_{1\leq k\leq Z_{n-1}}(\sigma^2 Z_{n-1})^{-1/2}|\sum_{j=1}^{k}(\xi_{n,j}-m)I(|(\xi-m)|>Z_{n-1}^{\frac{1}{2}})|>\frac{\alpha}{2}(2Ln)^{\frac{1}{2}}), && (5.46)\end{aligned}$$

where the last inequality follows because $Z_{n-1}>0$ on $B_{n,\alpha}$ implies

$$\begin{aligned} Z_{n-1}\frac{|\mu_{n,Z_{n-1}}|}{Z_{n-1}^{\frac{1}{2}}} &= Z_{n-1}^{\frac{1}{2}}|\int_{-Z_{n-1}^{\frac{1}{2}}}^{Z_{n-1}^{\frac{1}{2}}} t dF_{(\xi-m)}(t)| && (5.47)\\ &\leq Z_{n-1}^{\frac{1}{2}}\int_{Z_{n-1}^{\frac{1}{2}}}^{\infty} t dF_{|\xi-m|}(t) && (5.48)\\ &\leq \lim_{n\to\infty}\{\int_{Z_{n-1}^{\frac{1}{2}}}^{\infty} t^2 dF_{(\xi-m)}(t)+\int_{-\infty}^{-Z_{n-1}^{\frac{1}{2}}} t^2 dF_{(\xi-m)}(t) && (5.49)\\ &\leq \sigma^2\leq \frac{\alpha}{2}(2Ln)^{\frac{1}{2}} && (5.50)\end{aligned}$$

for all n sufficiently large. Thus,

$$\begin{aligned} II_n &\leq P(\max_{1\leq j\leq Z_{n-1}}|\xi_{n,j}-m|\geq Z_{n-1}^{\frac{1}{2}}) && (5.51)\\ &\leq \int_{Z_{n-1}>0} Z_{n-1}P(|\xi-m|\geq Z_{n-1}^{\frac{1}{2}}|Z_{n-1})dP && (5.52)\\ &\leq c(r,\xi)E((LZ_{n-1})^{-r}I(Z_{n-1}>0)), && (5.53)\end{aligned}$$

where the last inequality follows from (5.6) of Lemma 8. Thus $\sum_{n=1}^{\infty} II_n < \infty$ by using the harmonic moments result for LZ_n obtained in Appendix A.

We now deal with I_n. Since $1 \le \frac{\sigma}{\sigma_n} \le 2$ on $B_{n,\alpha}$ and $\lambda K^\alpha \supseteq K^\alpha$ for all $\lambda \ge 1$ we have

$$I_n \le P(\{\frac{\sigma}{\sigma_n}\frac{T_n}{(2Ln)^{\frac{1}{2}}} \notin K^\alpha\} \cap B_{n,\alpha}). \tag{5.54}$$

Hence since $B_{n,\alpha}$ is $\mathcal{F}_{n-1}$ measurable, the Markov property for $\{Z_n\}$ implies

$$I_n \le \int_{B_{n.\alpha}} P(\{\frac{\sigma}{\sigma_n}\frac{T_n}{(2Ln)^{\frac{1}{2}}} \notin K^\alpha\}|\mathcal{F}_{n-1})dP = \int_{B_{n,\alpha}} P(\{\frac{\sigma}{\sigma_n}T_n \notin (2Ln)^{1/2}K^\alpha\}|Z_{n-1})dP$$

Therefore, we have

$$I_n \le \int_{B_{n,\alpha}} [P(B \notin ((2Ln)^{1/2}K^\alpha)^{2b_n}) + 2b_n]dP,$$

where b_n denotes the Prokhorov metric distance between the law of T_n conditioned on Z_{n-1}, and the law of Wiener measure on $C_0[0,1]$, which we denote by writing

$$b_n = \rho(\mathcal{L}(\frac{\sigma}{\sigma_n}T_n|Z_{n-1}), \mathcal{L}(B)).$$

Now on $B_{n,\alpha}$ we have $Z_{n-1} > 0$, and if $2b_n < \frac{\alpha}{2}$ on $B_{n,\alpha}$, then Lemma 9 implies

$$I_n \le \int_{\{Z_{n-1}>0\}} (P(\frac{B}{(2Ln)^{\frac{1}{2}}} \notin K^{\frac{\alpha}{2}}) + 2b_n)dP. \tag{5.55}$$

To verify this last inequality we apply (5.7) of Lemma 8 and Corollary 2 of [12], with $c_E = c_5(3,1)$, to obtain

$$b_n \le c_E[c(r,\xi)/(LZ_{n-1})^r]^{1/4}, \tag{5.56}$$

and hence $2b_n < \alpha/2$ on $B_{n,\alpha}$ as required. Thus, $\sum_{n\ge1} I_n < \infty$ if

$$\sum_{n\ge1} P(\frac{B}{(2Ln)^{\frac{1}{2}}} \notin K^{\frac{\alpha}{2}}) < \infty, \tag{5.57}$$

and

$$\sum_{n\ge1} E((LZ_{n-1})^{-r/4} I(Z_{n-1} > 0)) < \infty. \tag{5.58}$$

To prove (5.57) we use Schilder's Theorem for Wiener measure ([10]), which implies for any $\delta > 0$ and $n \ge n(\delta)$ that

$$\log P(B \notin (2\log n)^{\frac{1}{2}} K^{\frac{\alpha}{2}}) \le -2\log n \Lambda((K^{\frac{\alpha}{2}})')(1-\delta),$$

where

$$\Lambda(A) = \inf\{\frac{||f||_\mu^2}{2} : f \in A\}, \ A \subset C_0[0,1].$$

Now, note that $\Lambda((K^{\frac{\alpha}{2}})') > \frac{(1+\beta)}{2}$ for some $\beta > 0$. Hence, for $n \ge n(\delta)$

$$\log P(B \notin (2\log n)^{\frac{1}{2}} K^{\frac{\alpha}{2}}) \le -(1+\beta)(1+\delta)\log n.$$

Now, choosing $\delta > 0$ such that $(1+\beta)(1-\delta) > 1 + \frac{\beta}{2}$, (5.57) holds. Finally, (5.58) follows by the harmonic moments result for LZ_n in Appendix A, and that $r/4 > 1$ in the above. This completes the proof of the lemma. ∎

Before we move on to establish (6.91) we introduce one more bit of a notation.

Definition 1. *Let* $\mathbf{h} \in (C_0[0,1])^\infty$ *and define*

$$||\mathbf{h}||^2_{\mu_\infty} = \sum_{j \geq 1} ||h_j||^2_\mu, \tag{5.59}$$

where $\mathbf{h} = (h_1, h_2, , h_3, \cdots)$*;also set*

$$J(\mathbf{f}, \delta) = \inf\{||\mathbf{h}||^2_{\mu_\infty} : \mathbf{h} \in (C_0[0,1])^\infty, d_\infty(\mathbf{f}, \mathbf{h}) < \delta\}. \tag{5.60}$$

Note that $||\mathbf{h}||^2_{\mu_\infty} = \infty$ on a dense subset of $(C_0[0,1])^\infty$, but that $J(\mathbf{f}, \delta) < \infty$ for all $\mathbf{f} \in (C_0[0,1])^\infty$ and all $\delta > 0$. Also, $\mathbf{f} \in (C_0[0,1])^\infty$ implies $f(0) = (0, 0, \cdots)$. Our next lemma is a key technical lemma needed in the proof of (6.91).

Lemma 11. *Let* $\mathbf{f} \in (C_0[0,1])^\infty$ *and assume that* $\mathbf{f} \notin K_\infty$*. Then there exists a* $\delta > 0$ *and an* $\eta(\delta) > 0$ *such that*

$$d_\infty(\mathbf{f}, K_\infty) > 3\delta, \tag{5.61}$$

and

$$J(\mathbf{f}, 2\delta) > 1 + 2\eta. \tag{5.62}$$

Proof. Note that since K_∞ is closed, and $\mathbf{f} \notin K_\infty$ (5.61) is obvious. Now, suppose (5.62) fails. Then, there exists a sequence $\{\mathbf{h}_n : n \geq 1\} \in (C_0[0,1])^\infty$ such that $d_\infty(\mathbf{h}_n, \mathbf{f}) < 2\delta$ for all $n \geq 1$ and

$$\lim_{n\to\infty} ||\mathbf{h}_n||^2_{\mu_\infty} \leq 1. \tag{5.63}$$

This implies that for every $\epsilon > 0$ there exists an $N(\epsilon)$ such that

$$\sup_{n \geq N(\epsilon)} ||\mathbf{h}_n||^2_{\mu_\infty} \leq 1 + \epsilon, \tag{5.64}$$

and hence $\{\mathbf{h}_n : n \geq N(\epsilon)\} \subset (1+\epsilon)K_\infty$. Now, since $(1+\epsilon)K_\infty$ is compact, and $(C_0[0,1])^\infty$ is separable, there exists a subsequence $\{\mathbf{h}_{n_k} : k \geq 1\}$ and $\mathbf{g} \in (C_0[0,1])^\infty$ such that $\lim_{k\to\infty} d_\infty(\mathbf{h}_{n_k}, \mathbf{g}) = 0$. Furthermore, $\mathbf{g} \in (1+\epsilon)K_\infty$ for all $\epsilon > 0$, by (5.64) and the fact that $(1+\epsilon)K_\infty$ is closed in $(C_0[0,1])^\infty$. This implies, $\mathbf{g} \in K_\infty$. This is a contradiction to $\lim_{n\to\infty} d_\infty(\mathbf{h}_{n_k}, \mathbf{g}) = 0$ since we have $\limsup_{n\to\infty} d_\infty(\mathbf{h}_n, \mathbf{f}) \leq 2\delta$ with $\delta > 0$ and $d_\infty(\mathbf{f}, K_\infty) > 3\delta$. ∎

Our next lemma establishes (6.91).

Lemma 12. *Under the conditions of the Theorem we have*

$$P(\lim_{n\to\infty} d_\infty(\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}, K_\infty) = 0) = 1. \tag{5.65}$$

Proof. By Lemma 9, the sequence $\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}\}$ is relatively compact in the complete separable metric space $((C_0[0,1])^\infty, d_\infty)$ with probability one. Thus, to prove (5.65) it is sufficient to show that for each $\mathbf{f} \notin K_\infty$, there exists an open subset V of $(C_0[0,1])^\infty$ containing 0 such that $(\mathbf{f}+V) \cap K_\infty = \phi$ and

$$P(\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}} \in \mathbf{f} + V \;\; i.o.) = 0. \tag{5.66}$$

Hence we turn to the proof of (5.66). Fix $\mathbf{f} \notin K_\infty$ and take $\delta > 0,\ \eta > 0$ as in Lemma 11. Then set $V = \prod V_j$, where $V_j = \alpha U$ and $U = \{f \in C_0[0,1] : ||f|| < 1\}$ for $1 \le j \le l$, $V_j = C_0[0,1]$ for $j \ge (l+1)$, and $l,\ \alpha$ are chosen to satisfy

$$3l\alpha + 3\sum_{j \ge l+1} 2^{-j} < \delta. \tag{5.67}$$

Then,

$$\mathbf{f} + 3V \subset \{\mathbf{g} : d_\infty(\mathbf{g}, \mathbf{f}) < 2\delta\} \tag{5.68}$$

and by our choice of $\delta > 0,\ \eta > 0$ we have by Lemma 10 that

$$J(\mathbf{f}, \delta) = \inf\{||\mathbf{h}||^2_{\mu_\infty} : \mathbf{h} \in \mathbf{f} + 3V\} > 1 + 2\eta. \tag{5.69}$$

Also observe that (5.61) implies that the closure of $\mathbf{f} + V$ does not intersect K_∞.

Let S denote the survival set and S_0 be the set as defined in (3.8). Furthermore, since $X_{n,Z_{n-1}} = 0$ eventually on S^c, we have that

$$P(\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{1/2}} \in \mathbf{f} + V\ \ i.o.\} \cap S^c) = 0. \tag{5.70}$$

Hence (5.66) holds from (5.70) and that $P(S\Delta S_0) = 0$ if we show that

$$P(\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}} \in \mathbf{f} + V\ \ i.o.\} \cap S_0) = 0. \tag{5.71}$$

Now, by definition of V, that $(\mathbf{f} + V) \cap K_\infty = \phi$, and that eventually $r(n) > l$, we see (5.71) holds if

$$P(\{\frac{\mathbf{X}_{n,l}}{(2Ln)^{\frac{1}{2}}} \in (f_1, \cdots, f_l) + (V_1 \times V_2 \times \cdots \times V_l)\ \ i.o.\} \cap S_0) = 0. \tag{5.72}$$

Let $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_n = \sigma(Z_1, \cdots, Z_n)$ for $n \ge 1$. Let

$$\mathcal{G}_{n,k} = \mathcal{F}_{nl+k}, k = 0, 1, 2\cdots l-1,\ n \ge 0, \tag{5.73}$$

and

$$E_n = \cap_{j=0}^{l-1} A_{n,j,\alpha}, \tag{5.74}$$

where

$$A_{n,j,\alpha} = \{\frac{X_{n-j,Z_{n-j-1}}}{(2Ln)^{\frac{1}{2}}} \in f_{j+1} + V_{j+1}\} \tag{5.75}$$

for $j = 0, 1, 2, \cdots l-1$. Then E_{nl+k} is $\mathcal{G}_{n,k}$ measurable and (5.72) holds by the conditional Borel-Cantelli lemma if we show that

$$\sum_{n \ge 1} P(E_{nl+k} | \mathcal{G}_{n-1,k}) < \infty \tag{5.76}$$

a.s. on S_0 for each $k = 0, 1, \cdots l-1$. That is, since $\{E_n\ i.o.\} \cap S_0$ is the event in (5.72) and

$$\{E_n\ \ i.o\} \cap S_0 \subseteq \cup_{k=0}^{l-1} \{E_{nl+k}\ \ i.o. \text{ in n}\} \cap S_0, \tag{5.77}$$

the conditional Borel-Cantelli lemma and (5.76) implies

$$P(\{E_{nl+k} \quad i.o. \text{ in n}\} \cap S_0) = 0. \tag{5.78}$$

Hence, (5.76) holding a.s. on S_0 for $k = 0, 1, 2, \cdots l-1$ and (5.77) and (5.78) combine to prove (5.72). We will prove (5.76) for $k = 0$ and observe that the other cases are exactly the same. Furthermore, to simplify out notation we will let $\mathcal{H}_n = \mathcal{G}_{n,0} = \mathcal{F}_{nl}$ for $n = 0, 1, \cdots$. Hence, we must show that

$$\sum_{n \geq 1} P(E_{nl}|\mathcal{H}_{n-1}) < \infty \tag{5.79}$$

a.s. on S_0.

To this end, since on S_0 we eventually have $Z_n > \beta^n$ for some $1 < \beta < m$ (see [1]), then for sufficiently large n we have

$$\begin{aligned} P(E_{nl}|\mathcal{H}_{n-1}) &= P(\cap_{j=0}^{l-1} A_{nl,j,\alpha}|\mathcal{H}_{n-1}) I(Z_{(n-1)l} > \beta^{(n-1)l}) & (5.80) \\ &= P(\cap_{j=0}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\})|\mathcal{H}_{n-1}), & (5.81) \end{aligned}$$

where the second equality holds since $Z_{(n-1)l}$ is $\mathcal{H}_{n-1}$ measurable. Thus, for all n sufficiently large, on S_0 we have

$$\begin{aligned} P(E_{nl}|\mathcal{H}_{n-1}) &= E(I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}) \cdot T_{n,l,\alpha}|\mathcal{H}_{n-1}\}) & (5.82) \\ &= \theta_{n,1} + \theta_{n,2}, & (5.83) \end{aligned}$$

where

$$T_{n,l,\alpha} = E(I(A_{nl,0,\alpha})|\mathcal{F}_{nl-1}), \tag{5.84}$$

$$\theta_{n,1} = E[I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)}\}) \cdot T_{n,l,\alpha,1}|\mathcal{H}_{n-1}], \tag{5.85}$$

$$\theta_{n,2} = E[I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)}\}) \cdot T_{n,l,\alpha,2}|\mathcal{H}_{n-1}], \tag{5.86}$$

$$T_{n,l,\alpha,1} = E(I(A_{nl,0,\alpha} \cap B_{n,\alpha})|\mathcal{F}_{nl-1}), \tag{5.87}$$

$$T_{n,l,\alpha,2} = E(I(A_{nl,0,\alpha} \cap B'_{n,\alpha})|\mathcal{F}_{nl-1}), \tag{5.88}$$

and

$$B_{n,\alpha} = \{c_E[\frac{c(r,\xi)}{(Z_{nl-1})^r}]^{1/4} < \rho_0, Z_{nl-1} > r_0(f_1, \cdots f_l; \alpha) \geq 1\}. \tag{5.89}$$

Here c_E is the constant from Corollary 2 of [12], $c(r, \xi)$ is given as in Lemma 8, and $Z_k > r_0(f_1, \cdots f_{l+1}; \alpha)$ implies $\frac{\sigma}{\sigma_k}(f_j + 2\alpha U) \subset (f_j + \frac{5}{2}\alpha U)$ for $j = 1, 2, \cdots l$, where $\sigma_k^2 = \sigma_{Z_k}^2$ is given as in (5.31). Note that the set $B_{n,\alpha}$ defined here is different from the one used previously, but it serves the same purpose in our calculation. Now

$$\theta_{n,2} \leq P(B'_{n,\alpha}|\mathcal{H}_{n-1}) \tag{5.90}$$

and hence

$$\sum_{n\geq 1} E(\theta_{n,2}) \leq \sum_{n\geq 1} P(B'_{n,\alpha}) < \infty \tag{5.91}$$

as in the argument yielding (5.36-37). Thus, $\sum_{n\geq 1}\theta_{n,2}$ converges with probability 1.

We now deal with $\theta_{n,1}$. If $Z_{n-1} > 0$, define $T_n(t)$ as in (5.39), and let $T_n(t) = 0$ for $0 \leq t \leq 1$ when $Z_{n-1} = 0$. Then recalling $V_1 = \alpha U$, we have

$$\begin{aligned} P(A_{nl,0,\alpha} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}) &= P(\{\frac{X_{nl,Z_{nl-1}}}{(2Lnl)^{\frac{1}{2}}} \in f_1 + V_1\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}) && (5.92)\\ &\leq I_n + II_n, && (5.93)\end{aligned}$$

where

$$I_n = P(\{\frac{T_{nl}}{(2Lnl)^{\frac{1}{2}}} \in (f_1 + 2\alpha U)\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}), \tag{5.94}$$

and

$$II_n = P(\{||X_{nl,Z_{nl-1}} - T_{nl}|| > \alpha(2Lnl)^{\frac{1}{2}}\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}). \tag{5.95}$$

Thus,

$$\begin{aligned} \theta_{n,1} &\leq E(I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)}\}) \cdot (I_n + II_n)|\mathcal{H}_{n-1}\}) && (5.96)\\ &\leq E(I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)}\}) \cdot I_n|\mathcal{H}_{n-1}\} + E(II_n|\mathcal{H}_{n-1}). && (5.97)\end{aligned}$$

We first deal with the second term. Arguing as we did following (5.44-45) and applying Lemma 8 we have

$$\begin{aligned} II_n &\leq Z_{nl-1} P(|\xi - m| \geq Z_{nl-1}^{\frac{1}{2}}|Z_{nl-1}) I(Z_{nl-1>0}) && (5.98)\\ &\leq c(r,\xi)(LZ_{nl-1})^{-r} I(Z_{nl-1} > 0). && (5.99)\end{aligned}$$

Thus by the harmonic moment results in Appendix A, we have $\sum_{n\geq 1} E(II_n|\mathcal{H}_{n-1}) < \infty$ a.s. on Ω.

We now turn to estimate I_n. To simplify writing, let

$$G_n = \cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}. \tag{5.100}$$

Hence on S_0 with $Z_{nl-1} \geq r_0(f_1 \cdots f_l; \alpha)$, we have $\frac{\sigma}{\sigma_{nl}}(f_1 + 2\alpha U) \subset f_1 + \frac{5}{2}\alpha U$. Hence,

$$\begin{aligned} I_n &= P(\{\frac{T_{nl}}{(2Lnl)^{\frac{1}{2}}} \in f_1 + 2\alpha U\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}) && (5.101)\\ &\leq P(\{\frac{\sigma}{\sigma_{nl}} \frac{T_{nl}}{(2Lnl)^{\frac{1}{2}}} \in (f_1 + \frac{5}{2}\alpha U\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}) && (5.102)\\ &\leq P(\frac{B}{(2Lnl)^{\frac{1}{2}}} \in f_1 + 3\alpha U) + b_n I(Z_{n-1} > 0), && (5.103)\end{aligned}$$

where the last inequality follows as in the argument used to obtain (5.55). That is, by definition of the Prokhorov metric $\rho(\cdot,\cdot)$, Corollary 2 of Einmahl, ([12]), and Lemma 8 we have

$$\begin{aligned} b_n &= \rho(\mathcal{L}(\frac{\sigma}{\sigma_{nl}} T_{nl}|Z_{nl-1}), \mathcal{L}(B)) && (5.104)\\ &\leq c_E[\frac{c(r,\xi)}{(LZ_{nl-1})^r}]^{1/4} < \rho_0 && (5.105)\end{aligned}$$

on $B_{n\alpha}$. Here $\mathcal{L}(B)$ is the law of the standard Brownian motion on $C_0[0,1]$, and we assume that $r_0(f_1,\cdots,f_l;\alpha)$ is sufficiently large to insure that ρ_0 above can be taken small enough that $\rho_0<\alpha/2$, which then implies for $\lambda\geq 1$ that $(\lambda(f_j+\frac{5}{2}\alpha U))^{\rho_0}\subset\lambda(f_j+3\alpha U)$ for $j=1,2,\cdots,l$. Thus,

$$\theta_{n,1}\leq\psi_{n,1}+\psi_{n,2}+II_n, \tag{5.106}$$

where

$$\psi_{n,1}=E(I(G_n|\mathcal{H}_{n-1})P(\frac{B}{(2Lnl)^{\frac{1}{2}}}\in(f_1+3\alpha U)) \tag{5.107}$$

and

$$\psi_{n,2}=c_Ec(r,\xi)^{1/4}E(I(G_n\cap B_{n,\alpha})(LZ_{nl-1})^{-r/4}|\mathcal{H}_{n-1}). \tag{5.108}$$

Now,

$$\sum_{n\geq 1}E(\psi_{n,2})\leq\sum_{n\geq 1}E((LZ_{nl-1})^{-r/4}I(Z_{nl-1>0}))<\infty \tag{5.109}$$

by the harmonic moment results of Appendix A and that $r>4$. Hence $\sum_{n\geq1}\psi_{n,2}$ converges with probability one and on S_0 we have

$$P(E_{nl}|\mathcal{H}_{n-1})\leq\psi_{n,1}+\psi_{n,3}, \tag{5.110}$$

where

$$\psi_{n,3}=\theta_{n,2}+\psi_{n,2}+II_n \tag{5.111}$$

and $\theta_{n,2}$, $\psi_{n,2}$, and II_n are summable with probability one. Now, recalling that G_n involves one less of the sets $A_{nl,j,\alpha}$, we iterate the above argument $l-1$ more times, starting at (5.80-83) with subsequent analogues of $B_{n,\alpha}$, to obtain

$$P(E_{nl}|\mathcal{H}_{n-1})\leq\{\prod_{j=0}^{l-1}P(\frac{B}{(2Lnl)^{\frac{1}{2}}}\in f_j+3\alpha U)I(Z_{(n-1)l}>\beta^{(n-1)l})\}+\psi_{n,4} \tag{5.112}$$

where $\sum_{n\geq1}\psi_{n,4}<\infty$. Now, using Schilder's large deviation estimate for Wiener measure as in [10], and our choice of α in forming the open set V, Lemma 11 implies for $\gamma>0$ and for all sufficiently large n that

$$P(E_{nl}|\mathcal{H}_{n-1})\leq\exp\{-\log_e(nl)(1-\gamma)(1+2\eta)\}I(Z_{(n-1)l}>\beta^{(n-1)l})+\psi_{n,4}. \tag{5.113}$$

The above estimate follows similarly to the proof of (5.57), where we now use lower bounds for the open set involved. Now taking γ sufficiently small so that $(1-\gamma)(1+2\eta)>1$, we have a.s. on S_0 that

$$\sum_{n\geq1}P(E_{nl}|\mathcal{H}_{n-1})<\infty. \tag{5.114}$$

The proof of the lemma now follows as indicated in (5.77), since the other $l-1$ cases are completely similar. ∎

Combining Lemmas 10 and 12, we see that with probability one we have

$$C(\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}\}) \subseteq K_\infty,$$

when we use the product topology on $(C_0[0,1])^\infty$. Our next lemma establishes that in the product topology the cluster set $C(\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}\})$ is actually K_∞ when we condition on non-extinction.

Lemma 13. *Under the assumptions of the Theorem, we have a.s. on S_0 that*

$$C(\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}\}) = K_\infty. \tag{5.115}$$

In particular, we have

$$P(C(\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}\}) = K_\infty|S_0) = 1.$$

Proof. Since the cluster set of a sequence of points in $((C_0[0,1])^\infty, d_\infty)$ is closed, and the topological space $((C_0[0,1])^\infty, d_\infty)$ is separable, it is easy to see that it is sufficient to show that for an arbitrary point $\mathbf{f} \in K_\infty$ with $||\mathbf{f}||_{\mu_\infty} < 1$, we have a.s. on S_0 that

$$\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}} \in \mathbf{f} + V \quad i.o. \tag{5.116}$$

where V is an arbitrary open set containing 0 of the form $V = \prod_{j\geq 1} V_j$ as above. The fact that we need not be concerned about points $\mathbf{f} \in K_\infty$ with $||\mathbf{f}||_{\mu_\infty} = 1$ follows since for such $\mathbf{f}$ we have $||(n-1)\mathbf{f}/n||_{\mu_\infty} < 1$ and $\lim_{\to\infty} d_\infty(\mathbf{f}, (n-1)\mathbf{f}/n) = 0$.

Thus by the conditional Borel Cantelli lemma it suffices to show that

$$\sum_{n\geq 1} P(E_{nl}|\mathcal{H}_{n-1}) = \infty \tag{5.117}$$

where E_{nl} and $\mathcal{H}_{n-1}$ are as before, except now $\mathbf{f} \in K_\infty$ with $||\mathbf{f}||_{\mu_\infty} < 1$, and l and α are arbitrary but fixed in our argument. Also let G_n be as given in (5.100). Then to verify (5.115), observe that for all n sufficiently large, on S_0

$$P(E_{nl}|\mathcal{H}_{n-1}) = E(E(I(A_{nl,0,\alpha}|\mathcal{F}_{nl-1})I(G_n)|\mathcal{H}_{n-1}) \tag{5.118}$$

$$> \theta_{n,1} - \theta_{n,2}, \tag{5.119}$$

where

$$\theta_{n,1} = E(I(G_n)E(I(A_{nl,0,\alpha} \cap B_{n,\alpha}|\mathcal{F}_{nl-1})|\mathcal{H}_{n-1}), \tag{5.120}$$

$$\theta_{n,2} \leq P(B'_{n,\alpha}|\mathcal{H}_{n-1}), \tag{5.121}$$

and

$$B_{n,\alpha} = \{c_E[c(r,\xi)(LZ_{nl-1})^{-r}]^{1/4} < \rho_0, Z_{nl-1} > r_0(f_1, \cdots f_l; \alpha) > 1\}, \tag{5.122}$$

where c_E and $c(r,\xi)$ are defined as above. Also, here we take $r_0(f_1, \cdots f_l, \alpha)$ such that $Z_k > r_0(f_1, \cdots f_l, \alpha)$ implies

$$\frac{\sigma}{\sigma_k}(f_j + \frac{3}{4}\alpha U) \supset (f_j + \frac{\alpha}{2}U) \tag{5.123}$$

for $j = 1, 2, \cdots, l$. From (5.90) and (5.91), $\sum_{n\geq 1} \theta_{n,2} < \infty$ a.s. on Ω, and arguing as in (5.92-95) and that $V_1 = \alpha U$,

$$\begin{aligned} P(A_{nl,0,\alpha} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}) &= P(\{X_{nl,Z_{nl-1}}/(2Lnl)^{\frac{1}{2}} \in (f_1 + \alpha U)\} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}) && (5.124) \\ &\geq I_n - II_n, && (5.125) \end{aligned}$$

where

$$I_n = P(\{\frac{T_{nl}}{(2Ln)^{\frac{1}{2}}} \in f_1 + \frac{3}{4}\alpha U\} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}), \tag{5.126}$$

and

$$II_n = P(\{||X_{nl,Z_{nl-1}} - T_{nl}|| > \frac{\alpha}{4}((2Lnl)^{\frac{1}{2}})\} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}). \tag{5.127}$$

Hence,

$$\begin{aligned} \theta_{n,1} &\geq E(I(G_n)(I_n - II_n) | \mathcal{H}_{n-1}) && (5.128) \\ &\geq E(I(G_n) I_n | \mathcal{H}_{n-1}) - E(II_n | \mathcal{H}_{n-1}), && (5.129) \end{aligned}$$

where as in (5.98-99), we have $\sum_{n\geq 1} E(II_n|\mathcal{H}_{n-1}) < \infty$ with probability one. Recalling the definition of $B_{n,\alpha}$ in this setting, we see that $Z_{nl-1} > r_0(f_1, f_2, \cdots f_l; \alpha)$ implies

$$\frac{\sigma}{\sigma_{nl}}(f_j + \frac{3}{4}\alpha U) \supseteq (f_j + \frac{\alpha}{2} U), \tag{5.130}$$

and hence

$$\begin{aligned} I_n &= P(\{\frac{T_{nl}}{(2Lnl)^{\frac{1}{2}}} \in f_1 + \frac{3}{4}\alpha U\} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}) && (5.131) \\ &\geq P(\{\frac{\sigma}{\sigma_{nl}} \frac{T_{nl}}{(2Lnl)^{\frac{1}{2}}} \in f_1 + \frac{1}{2}\alpha U | \mathcal{F}_{nl-1}\}) I(B_{n,\alpha}) && (5.132) \\ &\geq (P(\frac{B}{((2Lnl)^{\frac{1}{2}})} \in f_1 + \frac{\alpha}{4} U) - b_n) I(B_{n,\alpha}). && (5.133) \end{aligned}$$

Here the second inequality follows from Einmahl([12]) and Lemma 8, with b_n the Prokhorov distance between the conditional law $\mathcal{L}(\frac{\sigma}{\sigma_{nl}} T_{nl} | Z_{nl-1>0})$ and the law of standard Brownian motion on $C_0[0, 1]$. Furthermore, as before on $B_{n,\alpha}$ we have

$$b_n \leq c_E [c(r, \xi)(L Z_{nl-1})^{-r}]^{1/4} \leq \rho_0. \tag{5.134}$$

and for $\lambda \geq 1$ that $(\lambda(f_1 + \frac{\alpha}{4} U))^{\rho_0} \subseteq \lambda(f_1 + \frac{\alpha}{2} U)$ provided ρ_0 is sufficiently small that $\rho_0 < \alpha/4$. Now $\sum_{n\geq 1} E(b_n I(B_{n,\alpha})|\mathcal{H}_{n-1}) < \infty$ a.s. on Ω by the harmonic moment results of Appendix A ; i.e. $b_n < \rho_0$ implies $Z_{nl-1} > 0$, and we also have $\sum_{n\geq 1} \theta_{n,2} < \infty$ a.s. on Ω. Thus (5.128-133) imply that it suffices to show

$$\sum_{n\geq 1} P(G_n \cap B_{n,\alpha} | \mathcal{H}_{n-1}) P(\frac{B}{(2Lnl)^{\frac{1}{2}}} \in (f_1 + \frac{\alpha}{4} U)) = \infty \tag{5.135}$$

a.s. on S_0. Now,

$$P(G_n \cap B_{n,\alpha} | \mathcal{H}_{n-1}) = P(G_n | \mathcal{H}_{n-1}) - P(B'_{n,\alpha} | \mathcal{H}_{n-1}) \tag{5.136}$$

and since $\sum_{n\geq 1} P(B'_{n,\alpha}|\mathcal{H}_{n-1}) < \infty$ with probability one by what we did earlier, (5.117) will follow if we show that a.s. on S_0

$$\sum_{n\geq 1} P(G_n|\mathcal{H}_{n-1})P(\frac{B}{(2Lnl)^{\frac{1}{2}}} \in (f_1 + \frac{\alpha}{4}U)) = \infty. \tag{5.137}$$

Iterating the previous argument $l-1$ more times we see that it suffices to show that a.s. on S_0

$$\sum_{n\geq 1}\prod_{j=1}^{l} P(\frac{B}{(2Lnl)^{\frac{1}{2}}} \in (f_j + \frac{\alpha}{4}U))I(Z_{(n-1)l} > \beta^{(n-1)l}) = \infty. \tag{5.138}$$

Since

$$P(\{Z_{(n-1)l} > \beta^{(n-1)l} \ eventually\} \cap S_0) = P(S_0), \tag{5.139}$$

it is sufficient to show that

$$\sum_{n\geq 1}\prod_{j=1}^{l} P(\frac{B}{(2Lnl)^{\frac{1}{2}}} \in (f_j + \frac{\alpha}{4}U)) = \infty. \tag{5.140}$$

Now by Schilder's large deviation estimate for Wiener measure as in (5.113), for $\gamma > 0$

$$\prod_{j=1}^{l} P(\frac{B}{(2Lnl)^{\frac{1}{2}}} \in (f_j + \frac{\alpha}{4}U)) \geq \exp\{-2Lnl\sum_{j=1}^{l} ||f_j||_{\mu}^2(\frac{1}{2}(1-\frac{\gamma}{2})\}, \tag{5.141}$$

provided $n \geq n(\gamma)$ for some finite constant $n(\gamma)$. Thus for l fixed, by taking $n > \hat{n}(\gamma)$ sufficiently large we have

$$\prod_{j=1}^{l} P(\frac{B}{(2Lnl)^{\frac{1}{2}}} \in (f_j + \frac{\alpha}{4}U)) \geq \exp\{-Ln\sum_{j=1}^{l} ||f_j||_{\mu}^2(1-\gamma)\}. \tag{5.142}$$

Since in this case $||\mathbf{f}||_{\mu_\infty}^2 < 1$, we have $\sum_{j=1}^{l} ||f_j||_{\mu}^2 < 1$, and we can choose $\gamma > 0$ sufficiently small so that $\sum_{j=1}^{(l)} ||f_j||_{\mu}^2(1-\gamma) < 1$. Hence for all sufficiently large n we have,

$$\prod_{j=1}^{l} P(\frac{B}{(2Lnl)^{\frac{1}{2}}} \in (f_j + \frac{\alpha}{4}U)) \geq \frac{1}{n} \tag{5.143}$$

yielding (5.117). This proves (5.115) and the lemma is proven. ∎

Proof of Theorem 5. The proof of Theorem 5 follows by combining Lemma 10, Lemma 12 and Lemma 13.

Replacing the normalizer Z_{n-1} in the denominator of the definition of $X_{n,Z_{n-1}}(t)$ in (2.12) by m^{n-1} suggests the possibility that a result similar to Proposition 1 on the LLN might hold for our analogue of Strassen's theorem. For these purposes we define the $C_0[0,1]$-valued process

$$X_{n,Z_{n-1},m^{n-1}}(t) = (\frac{Z_{n-1}}{m^{n-1}})^{\frac{1}{2}} X_{n,Z_{n-1}}(t), \quad 0 \leq t \leq 1, \tag{5.144}$$

and also the $(C_0[0,1])^{\infty}$-valued process on $[0,1]$ given by

$$\mathbf{X}_{n,r(n),m^{n-1}}(t) = (X_{n,Z_{n-1},m^{n-1}}(t), \cdots, X_{n-r(n)+1,Z_{n-r(n)},m^{n-r(n)}}(t), 0, 0. \ldots). \tag{5.145}$$

Then the following holds.

Proposition 2. *Assume $E(Z_1^2(L(Z_1))^r) < \infty$ for some $r > 4$, that $1 \le r(n) \le n$, and $\lim_{n\to\infty} r(n) = \infty$. Let W be given as in Proposition 1. Then*

$$P(\lim_{n\to\infty} d_\infty(\frac{\mathbf{X}_{n,r(n),m^{n-1}}}{(2Ln)^{\frac{1}{2}}}, W^{\frac{1}{2}}K_\infty) = 0) = 1, \tag{5.146}$$

where the d_∞-distance from a point to a set is defined as usual. In addition, if S denotes the survival set of the process and clustering is determined with respect to the product topology, then we have

$$P(C(\{\frac{\mathbf{X}_{n,r(n),m^{n-1}}}{(2Ln)^{\frac{1}{2}}}\}) = W^{\frac{1}{2}}K_\infty|S) = 1. \tag{5.147}$$

Proof. First of all observe that

$$\mathbf{X}_{n,r(n),m^{n-1}}(t) = ((\frac{Z_{n-1}}{m^{n-1}})^{\frac{1}{2}}X_{n,Z_{n-1}}(t), \cdots, (\frac{Z_{n-r(n)}}{m^{n-r(n)}})^{\frac{1}{2}}X_{n-r(n)+1,Z_{n-r(n)}}(t), 0, 0, \ldots), \tag{5.148}$$

and Theorem 4 implies with probability one that

$$\lim_{n\to\infty} d_\infty(\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}, K_\infty) = 0. \tag{5.149}$$

Now recall that on S^c we have $X_{n,Z_{n-1}}(\cdot) = 0$ for all sufficiently large n and that $(0, 0, \cdots) \in K_\infty$. Hence it suffices to to show almost surely on S that

$$\lim_{n\to\infty} d_\infty(\frac{\mathbf{X}_{n,r(n),m^{n-1}}}{(2Ln)^{\frac{1}{2}}}, W^{\frac{1}{2}}K_\infty) = 0 \tag{5.150}$$

Letting $E = \{\omega \in S : \lim_{n\to\infty} \frac{Z_{n-1}}{m^{n-1}}(\omega) = W(\omega) < \infty\}$, we have $P(E \triangle S) = 0$. Now fix $\omega \in E$ such that (5.149) holds. Then the relative compactness of Lemma 10, or that K_∞ is compact and (5.149) holds, implies for every subsequence n' of the integers there exists a further subsequence n'' and $\mathbf{k}(\cdot,\omega) = (k_1(\cdot,\omega), k_1(\cdot,\omega), k_1(\cdot,\omega), \cdots) \in K_\infty$ such that

$$\lim_{n''\to\infty} d_\infty(\frac{\mathbf{X}_{n'',r(n'')}(\cdot,\omega)}{(2Ln'')^{\frac{1}{2}}}, \mathbf{k}(\cdot,\omega)) = 0, \tag{5.151}$$

which implies

$$\lim_{n''\to\infty} ||\frac{X_{n''-j+1,Z_{n''-j}}(\cdot,\omega)}{(2Ln'')^{\frac{1}{2}}} - k_j(\cdot,\omega)|| = 0 \tag{5.152}$$

for all $j \ge 1$. Furthermore, since $\omega \in E$, we also have

$$\lim_{n\to\infty} \frac{Z_{n-1}(\omega)}{m^{n-1}} = W(\omega) < \infty. \tag{5.153}$$

Now

$$\lim_{n\to\infty} d_\infty(\frac{\mathbf{X}_{n,r(n),m^{n-1}}(\cdot,\omega)}{(2Ln)^{\frac{1}{2}}}, W^{\frac{1}{2}}(\omega)K_\infty) = 0 \tag{5.154}$$

if and only if for each subsequence n' of the integers there is a further subsequence n'' and $\mathbf{h}(\cdot,\omega) \in K_\infty$ such that

$$\lim_{n''\to\infty} d_\infty(\frac{\mathbf{X}_{n'',r(n''),m^{n''-1}}(\cdot,\omega)}{(2Ln'')^{\frac{1}{2}}}, W^{\frac{1}{2}}(\omega)\mathbf{h}(\cdot,\omega)) = 0, \tag{5.155}$$

Furthermore, (5.155) holds if and only if for all $j \geq 1$

$$\lim_{n'' \to \infty} || \frac{X_{n''-j+1, Z_{n''-j}, m^{n''-j}}(\cdot, \omega)}{(2Ln'')^{\frac{1}{2}}} - W^{\frac{1}{2}}(\omega) h_j(\cdot, \omega)|| = 0. \tag{5.156}$$

Hence for $\omega \in E$ such that (5.149) holds, take a subsequence n' of the integers. Now take a further subsequence n'' and $\mathbf{k}(\cdot, \omega) \in K_\infty$ such that (5.151) holds. Also set $\mathbf{h}(\cdot, \omega) = \mathbf{k}(\cdot, \omega)$, where $\mathbf{k}(\cdot, \omega)$ is as in (5.151). Then

$$|| \frac{X_{n''-j+1, Z_{n''-j}, m^{n''-j}}(\cdot, \omega)}{(2Ln'')^{\frac{1}{2}}} - W^{\frac{1}{2}}(\omega) k_j(\cdot, \omega)|| \leq A_{n''} + B_{n''}, \tag{5.157}$$

where

$$A_{n'', j, \omega} = ||(\frac{Z_{n''-j}}{m^{n''-j}})^{\frac{1}{2}} [\frac{X_{n''-j+1, Z_{n''-j}}(\cdot, \omega)}{(2Ln'')^{\frac{1}{2}}} - k_j(\cdot, \omega)]||, \tag{5.158}$$

and

$$B_{n'', j, \omega} = ||k_j(\omega)[(\frac{Z_{n''-j}}{m^{n''-j}})^{\frac{1}{2}} - W^{\frac{1}{2}}(\omega)]||. \tag{5.159}$$

Given (5.152) for all $j \geq 1$, we thus have $\lim_{n'' \to \infty} A_{n'', j, \omega} = 0$ and $\lim_{n'' \to \infty} B_{n'', j, \omega} = 0$ for all $j \geq 1$ and $\omega \in E$, and hence (5.156) holds for all $j \geq 1$. Hence (5.155) holds with $\mathbf{h} = \mathbf{k}$, and (5.146) is proven.

To finish the proof of Proposition 2 one proves (5.147) by repeating the argument above arguing along suitable subsequences. That is, if $\mathbf{k}$ is in the cluster set

$$C(\{\frac{\mathbf{X}_{n, r(n)}}{(2Ln)^{\frac{1}{2}}}\}), \tag{5.160}$$

and $\omega \in E$, then $W^{\frac{1}{2}}(\omega)\mathbf{k}$ is in the cluster set

$$C(\{\frac{\mathbf{X}_{n, r(n), m^{n-1}}}{(2Ln)^{\frac{1}{2}}}\}). \tag{5.161}$$

Hence almost surely on S we have $C(\{\frac{\mathbf{X}_{n, r(n), m^{n-1}}}{(2Ln)^{\frac{1}{2}}}\}) = W^{\frac{1}{2}} K_\infty$, and Proposition 2 is proven. ■

6 The Chung-Wichura Functional Law of the Logarithm

Now we turn to the proof of the multiple generation functional law of Chung-Wichura type given in Theorem 5. Of course, Theorem 6 is a simple consequence of Theorem 5 by taking $r(n) = 1$, so we restrict our attention to Theorem 5.

The basic outline of the proof is much the same as for our version of Strassen's result obtained in Theorem 4, but the details are quite different. An immediate simplification in this case is that the infinite product space $(\mathcal{M}^\infty, \rho_\infty)$ is a compact metric space. This follows since the metric ρ_∞ gives the product topology on $\mathcal{M}^\infty$, and by the definition (2.25) the space $(\mathcal{M}, \rho)$ itself is a compact metric space, i.e. $(\mathcal{M}, \rho)$ is homeomorphic to the space of sub-probabilities on $[0, 1]$ under the Lévy metric, which is a compact metric space. Hence in order to prove Theorem 6 it suffices obtain analogues of Lemmas 12 and 13.

First, however, we show the limit set $\mathcal{K}^\infty$ is a compact subset of $(\mathcal{M}^\infty, \rho_\infty)$.

Lemma 14. *$\mathcal{K}^\infty$ is a compact subset of the space $(\mathcal{M}^\infty, \rho_\infty)$.*

Proof. Since $(\mathcal{M}^\infty, \rho_\infty)$ is compact, it suffices to show $\mathcal{K}^\infty$ is a closed subset of $(\mathcal{M}^\infty, \rho_\infty)$. Hence let $\{\mathbf{f}_n\}$ be a sequence in $\mathcal{K}^\infty$, and $\mathbf{f} \in \mathcal{M}^\infty$ such that $\mathbf{f}_n = (f_{n,1}, f_{n,2}, \cdots)$ and $\mathbf{f} = (f_1, f_2, \cdots)$. Then, since ρ_∞ gives the product topology on $\mathcal{M}^\infty$, we have $\lim_{n\to\infty} \rho_\infty(\mathbf{f}_n, \mathbf{f}) = 0$ iff $\lim_{n\to\infty} \rho(f_{n,j}, f_j) = 0$ for every $j \geq 1$. Furthermore, (2.25-26) and the classical facts regarding convergence in Levy's metric, see [15], pp 32-37, imply that $\lim_{n\to\infty} \rho(f_{n,j}, f_j) = 0$ iff $\lim_{n\to\infty} f_{n,j}(t) = f_j(t)$ for all $t \in [0,1]$ which are continuity points of the limit function f_j. Hence it is immediate that $\lim_{n\to\infty} \rho_\infty(\mathbf{f}_n, \mathbf{f}) = 0$ iff for all $j \geq 1$ we have

$$\lim_{n\to\infty} f_{n,j}(t) = f_j(t),$$

except possibly for countably many $t \in [0,1]$.

Hence let $\{\mathbf{f}_n\}$ be sequence in $\mathcal{K}_\infty$ with $\lim_{n\to\infty} \rho_\infty(\mathbf{f}_n, \mathbf{f}) = 0$. Then for every integer $n \geq 1$ the above implies we have that

$$\begin{aligned} \sum_{j=1}^{N} \int_0^1 f_j^{-2}(s)ds &= \sum_{j=1}^{N} \int_0^1 \lim_{n\to\infty} f_{n,j}^{-2}(s)ds && (6.1) \\ &\leq \liminf_{n\to\infty} \sum_{j=1}^{N} \int_0^1 f_{n,j}^{-2}(s)ds && (6.2) \\ &\leq 1, && (6.3) \end{aligned}$$

where the first inequality above is Fatou's lemma and the second because $\mathbf{f}_n \in \mathcal{K}_\infty$. Since N is arbitrary, this implies $\mathbf{f} \in \mathcal{K}_\infty$, so we have $\mathcal{K}_\infty$ is closed. Thus the lemma is proven. ∎

Now we introduce some further notation, which will yield a useful open neighborhood base for the topological space $(\mathcal{M}, \rho)$.

Definition. If $f \in \mathcal{M}$, then we set $t_f^* = \sup\{t : 0 \leq t \leq 1, f(t) < \infty\}$ and note that $t_f^* = 1$ by default if $f(1) < \infty$. If

$$0 = t_0 < t_1, t_2, \cdots, t_r < t_f^* \leq t_{r+1} < \cdots < t_{r+s} \leq 1$$

is an arbitrary partition of the interval $[0,1]$, we often will abbreviate the partition by $\mathcal{P}$ without explicitly displaying the points of the partition, or the number of points in the partition, which is also arbitrary. If $f \in \mathcal{M}$, $\alpha, \beta > 0$, and $\mathcal{P}$ is the partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_r < t_f^* \leq t_{r+1} < \cdots < t_{r+s} \leq 1$$

we define the neighborhood

$$N(f, t_1, t_2, \cdots, t_{r+s}, \alpha, \beta) = N^{(1)}(f, t_1, t_2, \cdots, t_r, \alpha) \cap N^{(2)}(f, t_{r+1}, t_{r+_2}, \cdots, t_{r+s}, \beta), \tag{6.4}$$

where

$$N^{(1)}(f, t_1, t_2, \cdots, t_r, \alpha) = \{g \in \mathcal{M} : f(t_j) - \alpha < g(t_j) < f(t_j) + \alpha,\ j = 1, \cdots, r\}$$

and

$$N^{(2)}(f, t_{r+1}, t_{r+_2}, \cdots, t_{r+s}, \beta) = \{g \in \mathcal{M} : g(t_{r+k}) > \beta,\ k = 1, \cdots, s\}.$$

When the partition $\mathcal{P}$ and α, β are understood we will sometimes simply write $N(f)$, $N(f, \mathcal{P})$, or $N(f, \mathcal{P}, \alpha, \beta)$. If the partition $\mathcal{P}$ contains only points from $[0, t_f^*)$, then we will use $N(f, \mathcal{P}, \alpha)$ to denote

$$\{g \in \mathcal{M} : f(t_j) - \alpha < g(t_j) < f(t_j) + \alpha,\ j = 1, \cdots, r\}.$$

Finally, if $t_f = 1$, $f(1) < \infty$, and $t = 1$ is a continuity point of f, then we will allow $t = 1$ in partitions of the form $N(f, \mathcal{P}, \alpha)$.

Our next lemma justifies the neighborhood terminology we use for the sets $N(f)$. Since $(\mathcal{M}, \rho)$ is homeomorphic to the space of sub-probabilities on $[0, 1]$ metrized by Lévy's metric, and convergence in Lévy's metric is equivalent to pointwise convergence at all points where the limit function is continuous, the same holds for ρ convergence on $\mathcal{M}$ by definition of ρ in (2.26). Hence the following lemma is hardly surprising, but we include the details for completeness.

Lemma 15. *The collection of sets $N(f, \mathcal{P}, \alpha, \beta)$, as $\mathcal{P}$ varies over all possible partitions of continuity points of f in $[0, 1]$ and we also allow $\alpha, \beta > 0$ to be arbitrary, forms an open neighborhood base at the point $f \in \mathcal{M}$. That is, given*

$$N(f, t_1, t_2, \cdots, t_{r+s}, \alpha, \beta) = N^{(1)}(f, t_1, t_2, \cdots, t_r, \alpha) \cap N^{(2)}(f, t_{r+1}, t_{r+_2}, \cdots, t_{r+s}, \beta),$$

there is an $\epsilon > 0$ and open neighborhood

$$H(f) = \{g \in \mathcal{M} : \rho(f, g) < \epsilon\}$$

such that $H(f) \subseteq N(f, t_1, t_2, \cdots, t_{r+s}, \alpha, \beta)$, and for each such $H(f)$ there is an $\alpha, \beta > 0$ and a partition $\mathcal{P}$ such that $N(f, \mathcal{P}, \alpha, \beta) \subseteq H(f)$. Moreover, if $f(1) < \infty$, then the sets $N(f, \mathcal{P}, \alpha)$, as $\mathcal{P}$ varies over all finite partitions of continuity points of f and $\alpha > 0$ is arbitrary, form an open neighborhood base at f.

Proof Once we prove the above set inclusions, the fact that sets $N(f)$ are actually open follows immediately since the inequalities that define $N(f)$ as per (6.4) are strict inequalities.

Hence we now show the set inclusions, starting with

$$N(f) = N^{(1)}(f, t_1, t_2, \cdots, t_r, \alpha) \cap N^{(2)}(f, t_{r+1}, t_{r+2}, \cdots, t_{r+k}, \beta)$$

as in (6.4) and finding $\epsilon > 0$ so that $H(f) \subseteq N(f)$. Thus it suffices to take $\epsilon > 0$ sufficiently small so that

$$\rho(f, g) = d_L(f^*, g^*) < \epsilon$$

implies $g \in N_f$, i.e. $|g(t_j) - f(t_j)| < \alpha$ for $1 \le j \le r$, and $g(t_{r+k}) > \beta$ for $k = 1, \cdots, s$.

Since $f \in \mathcal{M}$ is finite on $[0, t_f^*)$, then under the conditions on f imposed here, and $0 = t_0 < t_1 < \cdots < t_r < t_f^*$, we have f is uniformly bounded on $[0, T)$ by $f(T)$, where $t_r < T < t_f^*$. Next we observe that if $h(s) = s/(1 + s)$ for $0 \le s < \infty$, then h is continuous, strictly increasing, and has range $[0, 1)$. Hence h has an inverse h^{-1} which is continuous on $[0, 1)$. In particular, this implies that if $|h(b) - h(c)| \le \theta$ for $0 \le b, c \le f(T) + 1$, then for $\theta = \theta(\alpha) > 0$ sufficiently small we have $|b - c| < \alpha$. This is just the uniform continuity of h^{-1} on $[0, M/(1 + M)]$, where $M = f(T) + 1$.

Now $f^*(t_i) = f(t_i)/(1 + f(t_i) \in [0, f(t_r)/(1 + f(t_r)] \subseteq [0, M/(1 + M))$ for all $i = 1, \cdots, r$, and thus uniformly in $t_i, 1, \ldots, r$ if $|f^*(t_i) - g^*(t_i)| < \theta$ and $0 \le f^*(t_i), g^*(t_i) \le M/(1 + M)$ we have $|f(t_i) - g(t_i)| < \alpha$. Since $\alpha, \beta > 0$ are fixed we thus fix $\theta = \theta(\alpha, \beta) > 0$ so that the above holds, $0 < \theta = \theta(\alpha) < \{M/(1 + M) - f(T)/(1 + f(T))\}/2$, and also such that $(1 - \theta)/\theta > \beta$.

Using the continuity of f at each t_i we now choose $\epsilon > 0$ sufficiently small so that

$$f^*(t_i + \epsilon) - f^*(t_i - \epsilon) + 2\epsilon < \theta$$

uniformly for $i = 1, \cdots, r$. Here we recall that the f^* is defined to be zero on $(-\infty, 0]$ and $f^*(1)$ on $[1, \infty)$. Then $d(f, g) = d_L(f^*, g^*) < \epsilon$ implies

$$f^*(t_i - \epsilon) - \epsilon < g^*(t_i) < f^*(t_i + \epsilon) + \epsilon, 1 \leq i \leq r + s.$$

Hence since f^* is non-decreasing, $f^*(t_i - \epsilon) \leq f^*(t_i) \leq f^*(t_i + \epsilon)$, which implies

$$|f^*(t_i) - g^*(t_i)| < f^*(t_i + \epsilon) - f^*(t_i - \epsilon) + 2\epsilon < \theta$$

for $i = 1, \cdots, r + s$ and $0 \leq f^*(t_i), g^*(t_i) \leq M/(1 + M)$. Thus by our choice of θ, we have $g \in N^{(1)}(f, t_1, t_2, \cdots, t_r, \alpha) = \{g \in \mathcal{M} : f(t_j) - \alpha < g(t_j) < f(t_j) + \alpha,\ j = 1, \cdots, r\}$. In addition, since $f^*(t_r + k) = 1$ for $k = 1, \cdots, s$, then $g^*(t_{r+k}) > 1 - \epsilon > 1 - \theta$ by our choice of ϵ, and this implies $g(t_{r+k}) > \beta$ for $k = 1, \cdots, s$. Thus we also have $g \in N^{(2)}(f, t_{r+1}, t_{r+_2}, \cdots, t_{r+s}, \beta) = \{g \in \mathcal{M} : g(t_{r+k}) > \beta,\ k = 1, \cdots, s\}$, and hence $g \in N(f)$, which is what we want. Furthermore, speaking of N_f as a neighborhood of f is suitable terminology.

Next we fix $H(f) = \{g \in \mathcal{M} : \rho(f, g) < \epsilon\} = \{g \in \mathcal{M} : d_L(f^*, g^*) < \epsilon\}$. Thus

$$-\epsilon + \frac{f(t - \epsilon)}{1 + f(t - \epsilon)} < \frac{g(t)}{1 + g(t)} < \frac{f(t + \epsilon)}{1 + f(t + \epsilon)} + \epsilon, \tag{6.5}$$

for $t \geq 0$, where it is understood here that $f(t) = 0$ for $t \leq 0$ and $f(t) = f(1)$ for $t \geq 1$ when $f \in \mathcal{M}$.

Now we define suitable $N(f, \mathcal{P}, \alpha, \beta)$. To do this we take the partition $\mathcal{P}$ to be $0 = t_0 < t_1 < t_2 < \cdots, t_r < t_f^* \leq t_{r+1}, \cdots, t_{r+s}$ such that the points of the partition are all continuity points of f and $\max_{1 \leq j \leq r+s}(t_j - t_{j-1}) < \epsilon/2$. Next we choose α such that $0 < \alpha < \epsilon/2$ and $\beta > (1 - \epsilon)/\epsilon$.

Then since $h(s) = s/(1 + s)$ satisfies $|h(s) - h(t)| \leq |s - t|$ for $s, t \in [0, \infty]$, we thus have that $|f(t) - g(t)| < \alpha$ implies $|f^*(t) - g(t^*)| < \epsilon/2$ for all $t \in [0, \infty]$. Hence if $g \in N(f, \mathcal{P}, \alpha, \beta)$ and $t_{j-1} \leq t \leq t_j,\ 1 \leq j \leq r$, then

$$\frac{g(t)}{1 + g(t)} \leq \frac{g(t_j)}{1 + g(t_j)} \leq \frac{f(t_j)}{1 + f(t_j)} + \epsilon/2 \leq \frac{f(t + \epsilon)}{1 + f(t + \epsilon)} + \epsilon, \tag{6.6}$$

where the first and third inequalities are due to the fact that $h(s) = s/(1+s)$ is increasing on $[0, \infty]$ and the second follows since $|f(t_j) - g(t_j)| < \alpha$ implies $|f^*(t_j) - g^*(t_j)| < \epsilon/2$. An exactly similar argument implies

$$\frac{g(t)}{1 + g(t)} \geq \frac{g(t_{j-1})}{1 + g(t_{j-1})} \geq \frac{f(t_{j-1})}{1 + f(t_{j-1})} - \epsilon/2 \geq \frac{f(t - \epsilon)}{1 + f(t - \epsilon)} - \epsilon. \tag{6.7}$$

Similarly, if $t_r \leq t \leq t_f^* \leq t_{r+1}$, then $t + \epsilon > t_f^*$ and hence we have

$$\frac{g(t)}{1 + g(t)} \leq 1 \leq \frac{f(t + \epsilon)}{1 + f(t + \epsilon)} + \epsilon, \tag{6.8}$$

and also as in (6.7) that

$$\frac{g(t)}{1 + g(t)} \geq \frac{g(t_r)}{1 + g(t_r)} \geq \frac{f(t_r)}{1 + f(t_r)} - \epsilon/2 \geq \frac{f(t - \epsilon)}{1 + f(t - \epsilon)} - \epsilon. \tag{6.9}$$

Finally, if $t_{j-1} \le t \le t_j$ for $j \ge r+2$, then the inequality (6.8) is still valid, and we also have that

$$\frac{g(t)}{1+g(t)} \ge \frac{g(t_{j-1})}{1+g(t_{j-1})} > \frac{\beta}{1+\beta} > 1-\epsilon > \frac{f(t-\epsilon)}{1+f(t-\epsilon)} - \epsilon.$$

Thus we have for g in this $N(f, \mathcal{P}, \alpha, \beta)$, that $d_L(f^*, g^*) < \epsilon$, and hence $N(f, \mathcal{P}, \alpha, \beta) \subseteq H(f)$ as required. Of course, if $f(1) < \infty$, then the above shows there is an $N(f, \mathcal{P}, \alpha) \subseteq H(f)$.Thus the lemma is proven. ∎

Another elementary lemma involving the spaces $(\mathcal{M}, \rho)$ and $(\mathcal{M}^\infty, \rho_\infty)$ is as follows.

Lemma 16. *Let $f \in \mathcal{M}$ and for $n \ge 1$, $M > 0$ define*

$$h_n(t) = \frac{n+1}{n} f(t) \quad \text{and} \quad \mathrm{f^{(M)}(t) = f(t) \wedge M, \quad 0 \le t \le 1.}$$

Then

$$\rho(h_n, f) \le 1/n \quad \textit{and} \quad \rho(f^{(M)}, f) \le 1/(M+1).$$

Moreover, if $f \in \mathcal{K}$, then

$$\int_0^1 h_n^{-2}(s)ds = (n/(n+1))^2 \int_0^1 f^{-2}(s)ds < 1.$$

Furthermore, if $\mathbf{f} = (f_1, f_2, \cdots) \in \mathcal{K}_\infty$, $\mathbf{h}_n = \frac{n+1}{n}\mathbf{f}$, $\mathbf{f}^{(M)} = (f_1^{(M)}, f_2^{(M)}, \cdots)$, then

$$\rho_\infty(\mathbf{h}_n, \mathbf{f}) \le 1/n \quad \textit{and} \quad \rho_\infty(\mathbf{f}^{(M)}, \mathbf{f}) \le 1/(M+1),$$

and

$$\sum_{j=1}^{\infty} \int_0^1 h_n^{-2}(s)ds \le (\frac{n}{n+1})^2 < 1.$$

Proof. First observe that

$$\rho(h_n, f) = d_L(h_n^*, f^*) \le ||h_n^* - f^*||,$$

where the equality is by definition of the ρ-metric, and the inequality follows since the sup-norm dominates the Lévy metric. However, $h_n^*(t) - f^*(t) = 0$ if $t \ge t_f^*$ or $t = 0$, and for $0 < t < t_f^*$

$$h_n^*(t) - f^*(t) = \frac{f(t)/n}{(1+f(t))^2} \le 1/n.$$

Thus $||h_n - f|| \le 1/n$, which implies $\rho(h_n, f) \le 1/n$ as indicated. Similarly, $f^*(t) - (f^{(M)})^*(t) = 0$ if $0 \le f(t) \le M$ and for $M < f(t)$

$$f^*(t) - (f^{(M)})^*(t) = \frac{f(t)}{1+f(t)} - \frac{M}{1+M} \le 1 - \frac{M}{1+M} = \frac{1}{1+M}.$$

Thus $\rho(f^{(M)}, f) \le 1/(M+1)$ as indicated. The remainder of the proof is now immediate. ∎

To prove Theorem 5 we next prove a lemma which allows us to transfer estimates on $X_{n,Z_{n-1}}$ being close to B in law, to estimates on $M_{n,Zn-1}$ being close to

$$M_B(t) = \sup_{0 \le s \le t} |B(s)|, \quad 0 \le t \le 1 \tag{6.10}$$

in law.

Lemma 17. *Let $\Lambda : C[0,1] \to C[0,1]$ be defined by*

$$(\Lambda f)(t) = \sup_{0 \le s \le t} |f(s)|, \quad 0 \le t \le 1,$$

and for any Borel probability measure μ on $C[0,1]$ define $\mu^{\Lambda}(A) = \mu(\Lambda^{-1}(A))$ for Borel sets A. If $\rho(\mu,\nu)$ is the Prokhorov metric for probability measures on $C[0,1]$ when we use the sup-norm distance on $C[0,1]$, then

$$\rho(\mu^{\Lambda}, \nu^{\Lambda}) \le \rho(\mu, \nu). \tag{6.11}$$

Proof. Take $\delta > \rho(\mu,\nu)$ and A an arbitrary Borel subset of $C[0,1]$. Then we have

$$\mu^{\Lambda}(A) = \mu(\Lambda^{-1}(A)) \le \nu((\Lambda^{-1}(A))^{\delta}) + \delta \le \nu(\Lambda^{-1}(A^{\delta})) + \delta = \nu^{\Lambda}(A^{\delta}) + \delta. \tag{6.12}$$

In the above, the second inequality follows from the fact that Λ is a Lip-1 map with Lipshitz constant one, and hence $(\Lambda^{-1}(A))^{\delta} \subseteq \Lambda^{-1}(A^{\delta})$ for every A and $\delta > 0$. That is, if $f \in (\Lambda^{-1}(A))^{\delta}$, then there exists $g \in \Lambda^{-1}(A)$ with $||f-g||_{\infty} < \delta$. Hence $\Lambda(g) \in A$ and since $||\Lambda(f) - \Lambda(g)||_{\infty} \le ||f-g||_{\infty}$, this implies $\Lambda(f) \in A^{\delta}$ and hence $f \in \Lambda^{-1}(A^{\delta})$. The proof that $||\Lambda(f) - \Lambda(g)||_{\infty} \le ||f-g||_{\infty}$ follows easily from the triangle inequality. Finally (6.12) for arbitrary A and $\delta > \rho(\mu,\nu)$ implies the lemma.

Our next lemma proves convergence to the set $\mathcal{K}_{\infty}$ on the survival set S.

Lemma 18. *Let S denote the survival set of the process and set $c_2 = \pi^2/8$. Then, under the conditions of the theorem we have*

$$P(\{\lim_{n\to\infty} \rho_{\infty}((Ln/c_2)^{1/2}\mathbf{M}_{n,r(n)}, \mathcal{K}_{\infty}) = 0\} \cap S) = P(S). \tag{6.13}$$

Proof. To simplify notation we let

$$\eta_n(t) = (Ln/c_2)^{1/2} M_{n,Z_{n-1}}(t),\ 0 \le t \le 1,\ n \ge 1. \tag{6.14}$$

We also define the vector valued processes

$$\boldsymbol{\eta}_{n,l}(t) = (Ln/c_2)^{1/2}(M_{n,Z_{n-1}}(t), \cdots, M_{n-l+1,Z_{n-l}}(t)),\ 0 \le t \le 1,\ n \ge 1,\ l \ge 1, \tag{6.15}$$

and

$$\boldsymbol{\eta}_{n,r(n)}(t) = (Ln/c_2)^{1/2}(M_{n,Z_{n-1}}(t), \cdots, M_{n-r(n)+1,Z_{n-r(n)}}(t), 0, 0, \cdots),\ 0 \le t \le 1,\ n \ge 1. \tag{6.16}$$

Since $(\mathcal{M}_{\infty}, \rho_{\infty})$ is a compact metric space, it is separable. Hence $\mathcal{K}_{\infty}$ closed implies that (6.13) will follow if we show for every $\mathbf{f} \notin \mathcal{K}_{\infty}$ there exists an open set V containing $\mathbf{f}$ such that $V \cap \mathcal{K}_{\infty} = \phi$ and V satisfies

$$P(\{\boldsymbol{\eta}_{n,r(n)} \in V \ i.o.\} \cap S) = 0. \tag{6.17}$$

Letting S_0 be defined as in (3.8), we have $P(S \triangle S_0) = 0$, and hence it suffices to show

$$P(\{\boldsymbol{\eta}_{n,r(n)} \in V \ i.o.\} \cap S_0) = 0. \tag{6.18}$$

for each $\mathbf{f} \notin \mathcal{K}_{\infty}$ and suitable open set V disjoint from $\mathcal{K}_{\infty}$ containing $\mathbf{f}$.

If $\mathbf{f} = (f_1, f_2 \ldots) \notin \mathcal{K}_\infty$, then $\sum_{j\geq 1} \int_0^1 f_j^{-2}(s)ds > 1$. Hence there exists an integer $l \geq 1$ and $\delta > 0$ such that

$$\sum_{j=1}^{l} \int_0^1 f_j^{-2}(s)ds > 1 + \delta.$$

Furthermore, since the $f_j's$ are nondecreasing on $[0,1]$, there exist finite partitions $\mathcal{P}_j$ of $[0, t^*_{f_j})$ consisting of continuity points of f_j and $\alpha > 0$ such that

$$\sum_{j=1}^{l} \sum_{t_k \in \mathcal{P}_j} (f_j(t_k) + 4\alpha)^{-2}(t_k - t_{k-1}) > 1 + \delta. \tag{6.19}$$

Here the reader should note that we need not take any points in the partition $\mathcal{P}_j$ which are in $[t_{f_j^*}, 1]$ since $\int_{t_{f_j^*}}^1 f_j^{-2}(s)ds = 0$. In particular, if $t_{f_j^*} = 0$ we will not form a partition, but rather define $V_j = \mathcal{M}$, to be used as indicated below. That is, if $V = \prod_{j=1}^\infty V_j$, where $V_j = N(f_j, \mathcal{P}_j, \alpha)$, or $V_j = \mathcal{M}$ should $t_{f_j^*} = 1$, for $1 \leq j \leq l$, and $V_j = \mathcal{M}$ for $j \geq l+1$, then for $\mathbf{g} = (g_1, g_2, \cdots) \in V$ we have

$$\sum_{j\geq 1} \int_0^1 g_j^{-2}(s)ds \geq \sum_{j=1}^{l} \sum_{t_k \in \mathcal{P}_j} (f_j(t_k) + \alpha)^{-2}(t_k - t_{k-1}) > 1 + \delta. \tag{6.20}$$

Of course, if $t_{f_j^*} = 0$ for some j, $1 \leq j \leq l$, then those terms do not need to appear in (6.20), but to simplify the notation we write the proof as if all $t_{f_j^*} > 0$ for $j = 1, \cdots, l$.

In particular, we now have $\mathbf{f} \in V$ and $V \cap \mathcal{K}_\infty = \phi$. Furthermore, since $V_j = \mathcal{M}$ for all $j \geq l+1$ and eventually $r(n) > l$, we have

$$\{\boldsymbol{\eta}_{n,r(n)} \in V\} = \{\boldsymbol{\eta}_{n,l} \in \prod_{j=0}^{l-1} V_{j+1}\} = \bigcap_{j=0}^{l-1} \{\eta_{n-j, Z_{n-j-1}} \in V_{j+1}\}. \tag{6.21}$$

Hence (6.18) will follow if we show

$$P(\{\boldsymbol{\eta}_{n,l} \in \prod_{j=0}^{l-1} V_{j+1} \ i.o.\} \cap S_0) = 0. \tag{6.22}$$

Letting $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_n = \sigma(Z_1, \cdots, Z_n)$ for $n \geq 1$, we define

$$\mathcal{G}_{n,k} = \mathcal{F}_{nl+k}, k = 0, 1, 2 \cdots l-1, \ n \geq 0, \tag{6.23}$$

and

$$E_n = \bigcap_{j=0}^{l-1} A_{n,j,\alpha}, \tag{6.24}$$

where

$$A_{n,j,\alpha} = \{\eta_{n-j, Z_{n-j-1}} \in V_{j+1} = N(f_{j+1}, \mathcal{P}_{j+1}, \alpha)\} \tag{6.25}$$

for $j = 0, 1, 2, \cdots l-1$. Strictly speaking these sets also involve δ through (6.19), but we supress that as our choice of α implies (6.19).

Then E_{nl+k} is $\mathcal{G}_{n,k}$ measurable and (6.22) holds by the conditional Borel-Cantelli lemma if we show that

$$\sum_{n\geq 1} P(E_{nl+k}|\mathcal{G}_{n-1,k}) < \infty \tag{6.26}$$

a.s. on S_0 for each $k = 0, 1, \cdots l-1$. That is, since $\{E_n \ i.o.\} \cap S_0$ is the event in (6.22) and

$$\{E_n \ \ i.o\} \cap S_0 \subseteq \cup_{k=0}^{l-1} \{E_{nl+k} \ \ i.o. \text{ in n}\} \cap S_0, \tag{6.27}$$

the conditional Borel-Cantelli lemma and (6.26) implies

$$P(\{E_{nl+k} \ \ i.o. \text{ in n}\} \cap S_0) = 0. \tag{6.28}$$

Hence, (6.26) holding a.s. on S_0 for $k = 0, 1, 2, \cdots l-1$ and (6.27) and (6.28) combine to prove (6.22). We will prove (6.26) for $k = 0$ and observe that the other cases are exactly the same. Furthermore, to simplify out notation we will let $\mathcal{H}_n = \mathcal{G}_{n,0} = \mathcal{F}_{nl}$ for $n = 0, 1, \cdots$. Hence, we must show that

$$\sum_{n\geq 1} P(E_{nl}|\mathcal{H}_{n-1}) < \infty \tag{6.29}$$

a.s. on S_0.

To this end, notice that on S_0 we have $Z_n > \beta^n$ for some $1 < \beta < m$ (see [1]). Then for sufficiently large n we have that

$$\begin{aligned} P(E_{nl}|\mathcal{H}_{n-1}) &= P(\cap_{j=0}^{l-1} A_{nl,j,\alpha}|\mathcal{H}_{n-1}) I(Z_{(n-1)l} > \beta^{(n-1)l}) && (6.30)\\ &= P(\cap_{j=0}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\})|\mathcal{H}_{n-1}), && (6.31)\end{aligned}$$

since $Z_{(n-1)l}$ is $\mathcal{H}_{n-1}$ measurable. Thus, for all n sufficiently large, on S_0 we have

$$\begin{aligned} P(E_{nl}|\mathcal{H}_{n-1}) &= E(I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}) \cdot T_{n,l,\alpha}|\mathcal{H}_{n-1}\}) && (6.32)\\ &= \theta_{n,1} + \theta_{n,2}, && (6.33)\end{aligned}$$

where

$$T_{n,l,\alpha} = E(I(A_{nl,0,\alpha})|\mathcal{F}_{nl-1}), \tag{6.34}$$

$$\theta_{n,1} = E[I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)}\}) \cdot T_{n,l,\alpha,1}|\mathcal{H}_{n-1}], \tag{6.35}$$

$$\theta_{n,2} = E[I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)}\}) \cdot T_{n,l,\alpha,2}|\mathcal{H}_{n-1}], \tag{6.36}$$

$$T_{n,l,\alpha,1} = E(I(A_{nl,0,\alpha} \cap B_{n,\alpha})|\mathcal{F}_{nl-1}), \tag{6.37}$$

$$T_{n,l,\alpha,2} = E(I(A_{nl,0,\alpha} \cap B'_{n,\alpha})|\mathcal{F}_{nl-1}), \tag{6.38}$$

and

$$B_{n,\alpha} = \{c_E[\frac{c(r,\xi)}{(Z_{nl-1})^r}]^{1/4} < \rho_0, Z_{nl-1} > r_0(f_1, \cdots f_l; \alpha) \geq 1\}. \tag{6.39}$$

Here c_E is the constant from Corollary 2 of [12], $c(r,\xi)$ is given as in Lemma 8, and $Z_k > r_0(f_1,\cdots f_{l+1};\alpha)$ implies $\frac{\sigma}{\sigma_k}N(f_j,\mathcal{P}_j,2\alpha) \subseteq N(f_j,\mathcal{P}_j,\frac{5}{2}\alpha)$ for $j=1,2,\cdots l$, where $\sigma_k^2 = \sigma_{Z_k}^2$ is given as in (5.31). Note that the set $B_{n,\alpha}$ defined here is different from the one used previously, but it serves the same purpose in our calculation. Now

$$\theta_{n,2} \leq P(B'_{n,\alpha}|\mathcal{H}_{n-1}) \tag{6.40}$$

and hence

$$\sum_{n\geq 1} E(\theta_{n,2}) \leq \sum_{n\geq 1} P(B'_{n,\alpha}) < \infty \tag{6.41}$$

as in the argument yielding (5.36-37). Thus $\sum_{n\geq 1}\theta_{n,2}$ converges with probability 1.

We now deal with $\theta_{n,1}$. If $Z_{n-1}>0$, define $T_n(t)$ as in (5.39), and let $T_n(t)=0$ for $0\leq t\leq 1$ when $Z_{n-1}=0$. Then recalling $V_1 = N(f_1,\mathcal{P}_1,\alpha)$ and the map Λ from Lemma 17, we have

$$\begin{aligned} P(A_{nl,0,\alpha}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}) &= P(\{\eta_{nl,Z_{nl-1}}\in V_1\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}) &(6.42)\\ &\leq I_n + II_n, &(6.43)\end{aligned}$$

where

$$I_n = P(\{(L(nl)/c_2)^{1/2}\Lambda(T_{nl})\in N(f_1,\mathcal{P}_1,2\alpha)\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}), \tag{6.44}$$

and

$$II_n = P(\{||\eta_{nl,Z_{nl-1}} - (L(nl)/c_2)^{1/2}\Lambda(T_{nl})|| > \alpha/2\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}). \tag{6.45}$$

Thus,

$$\begin{aligned} \theta_{n,1} &\leq E(I(\cap_{j=1}^{l-1}A_{nl,j,\alpha}\cap\{Z_{(n-1)l}>\beta^{(n-1)}\})\cdot(I_n+II_n)|\mathcal{H}_{n-1}\}) &(6.46)\\ &\leq E(I(\cap_{j=1}^{l-1}A_{nl,j,\alpha}\cap\{Z_{(n-1)l}>\beta^{(n-1)}\})\cdot I_n|\mathcal{H}_{n-1}\} + \Gamma_n, &(6.47)\end{aligned}$$

where $\Gamma_n = E(I(\{Z_{(n-1)l}>\beta^{(n-1)}\})II_n|\mathcal{H}_{n-1})$.

We first deal with the second term Γ_n. First we observe that

$$I(\{Z_{(n-1)l}>\beta^{(n-1)}\})II_n \leq \alpha_n+\beta_n,$$

where

$$\alpha_n = P((\frac{L(nl)}{c_2})^{\frac{1}{2}} \sup_{1\leq k\leq Z_{nl-1}} |\sum_{j=1}^{k}(\xi_{n,j}-m)I(|\xi_{n,j}-m|>Z_{nl-1}^{\frac{1}{2}})| > 0|\mathcal{F}_{nl-1}),$$

and

$$\beta_n = P(\{(\frac{L(nl)}{c_2})^{\frac{1}{2}}Z_{nl-1}|\mu_{n,Z_{nl-1}}|/Z_{nl-1}^{\frac{1}{2}} > \alpha/2\}\cap B_{n,\alpha}\cap\{Z_{(n-1)l}>\beta^{(n-1)}\}|\mathcal{F}_{nl-1}).$$

Applying Lemma 8 we thus have

$$\begin{aligned} \alpha_n &\leq Z_{nl-1}P(|\xi-m|\geq Z_{nl-1}^{\frac{1}{2}}|Z_{nl-1})I(Z_{nl-1>0}) &(6.48)\\ &\leq c(r,\xi)(LZ_{nl-1})^{-r}I(Z_{nl-1}>0). &(6.49)\end{aligned}$$

Thus by the harmonic moment results in Appendix A, we have $\sum_{n\geq 1}E(\alpha_n|\mathcal{H}_{n-1})<\infty$ a.s. on Ω.

We now turn to an estimate of β_n. When $Z_{nl-1} > 0$, arguing as in (5.47-50) we see

$$Z_{nl-1}|\mu_{n,Z_{nl-1}}|/Z_{nl-1}^{\frac{1}{2}} \leq \int_{Z_{nl-1}^{\frac{1}{2}}}^{\infty} t^2(Lt)^r dF_{|\xi-m|}(t)/(LZ_{nl-1})^r \leq \frac{c}{(LZ_{nl-1})^r}$$

for some finite constant c since $E(\xi^2(L\xi)^r) < \infty$. Hence

$$\beta_n \leq P(\{(\frac{L(nl)}{c_2})^{\frac{1}{2}}\frac{c}{(LZ_{nl-1})^r} > \alpha/2\} \cap \{Z_{nl-1} > 3\} \cap \{Z_{(n-1)l} > \beta^{(n-1)}\}|\mathcal{F}_{nl-1}),$$

which implies

$$\beta_n \leq P(\{3 \leq Z_{nl-1} \leq \exp\{u(L(nl))^{\frac{1}{2r}}\}\} \cap \{Z_{(n-1)l} > \beta^{(n-1)}\}|\mathcal{F}_{nl-1}),$$

where u is a finite positive constant depending only on α, c, r, c_2. Letting $H_n = \{3 \leq Z_{nl-1} \leq \exp\{u(L(nl))^{\frac{1}{2r}}\}$ and $B_k = \{Z_{(n-1)l} = k\}$ for $k = 0, 1, 2, \cdots$, then since these sets are $\mathcal{F}_{nl-1}$ measurable we have

$$\beta_n \leq I(H_n)I(\{Z_{(n-1)l} > \beta^{(n-1)}\}).$$

Using the Markov property we have

$$E(\beta_n|\mathcal{H}_{n-1}) \leq E(I(H_n)|Z_{(n-1)l})I(\{Z_{(n-1)l} > \beta^{(n-1)}\}).$$

Now

$$E(I(H_n)|Z_{(n-1)l})I(\{Z_{(n-1)l} > \beta^{(n-1)}\}) = \sum_{k\geq 0} \frac{\int_{B_k} I(H_n)dP}{P(B_k)} I(B_k)I(\{Z_{(n-1)l} > \beta^{(n-1)}\})$$

$$= \sum_{k\geq[\beta^{(n-1)l}]+1} I(B_k)\frac{P(\{Z_{(n-1)l} = k\} \cap H_n)}{P(B_k)}$$

$$= \sum_{k\geq[\beta^{(n-1)l}]+1} I(B_k)P(H_n|Z_{(n-1)l} = k),$$

and hence we have

$$E(\beta_n|\mathcal{H}_{n-1}) \leq \sum_{k\geq[\beta^{(n-1)l}]+1} P(3 < \sum_{j=1}^{k} Z_{l-1,j} < \exp\{u(L(nl))^{\frac{1}{2r}}\}),$$

where $\mathcal{L}(Z_{l-1,j}) = \mathcal{L}(Z_{l-1})$ are independent for $j \geq 1$. Hence

$$E(\beta_n|\mathcal{H}_{n-1}) \leq \sum_{k\geq[\beta^{(n-1)l}]+1} P(\exp\{-\sum_{j=1}^{k} Z_{l-1,j}\} \geq \exp\{-\exp\{u(L(nl))^{\frac{1}{2r}}\}\}) \leq,$$

and Markov's inequality therefore implies

$$E(\beta_n|\mathcal{H}_{n-1}) \leq \sum_{k\geq[\beta^{(n-1)l}]+1} \exp\{\exp\{u(L(nl))^{\frac{1}{2r}}\}\}(E(\exp\{-Z_{l-1}\}))^k)$$

$$= \sum_{k\geq[\beta^{(n-1)l}]+1} \exp\{\exp\{u(L(nl))^{\frac{1}{2r}}\}\}\gamma^k = \exp\{\exp\{u(L(nl))^{\frac{1}{2r}}\}\}\frac{\gamma^{[\beta^{(n-1)l}]+1}}{1-\gamma},$$

where $\gamma = E(\exp\{-Z_{l-1}\}) < 1$ since $p_0 < 1$. Since $r > 4$ we have $1/2r < 1$, and thus for large n we have

$$\exp\{\exp\{u(L(nl))^{\frac{1}{2r}}\}\}\frac{\gamma^{[\beta^{(n-1)l}]+1}}{1-\gamma} \leq \gamma^{\frac{1}{2}\beta^{(n-1)l}}.$$

Thus for such n we have

$$E(\beta_n|\mathcal{H}_{n-1}) \leq \gamma^{\frac{1}{2}\beta^{(n-1)l}},$$

and hence $\sum_{n\geq 1} E(\beta_n|\mathcal{H}_{n-1}) < \infty$ almost surely, which implies $\sum_{n\geq 1}\Gamma_n = \sum_{n\geq 1} E(\alpha_n+\beta_n|\mathcal{H}_{n-1})$ converges with probability one on Ω.

We now turn to estimating I_n. To simplify writing, let

$$G_n = \cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}. \tag{6.50}$$

Hence on S_0 with $Z_{nl-1} \geq r_0(f_1\cdots f_l;\alpha)$, we have $\frac{\sigma}{\sigma_{nl}}N(f_1,\mathcal{P}_1,2\alpha) \subset N(f_1,\mathcal{P}_1,\frac{5}{2}\alpha)$. Hence,

$$\begin{aligned}
I_n &= P(\{\Lambda(T_{nl}) \in (c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,2\alpha)\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}) && (6.51)\\
&\leq P(\{\Lambda(\frac{\sigma}{\sigma_{nl}}T_{nl}) \in (c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,\frac{5}{2}\alpha)\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}) && (6.52)\\
&\leq P(M_B \in [(c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,\frac{5}{2}\alpha)]^{2b_n}) + 2b_nI(B_{n,\alpha}), && (6.53)
\end{aligned}$$

where the last inequality follows as in the argument used to obtain (5.55), Lemma 17, M_B is as defined in (6.10), and b_n denotes the Prokhorov distance $\rho(\mathcal{L}(\frac{\sigma}{\sigma_{nl}}T_{nl}|Z_{nl-1}),\mathcal{L}(B))$. Furthermore, Corollary 2 of Einmahl, ([12]), and Lemma 8 implies we have

$$\begin{aligned}
b_n &= \rho(\mathcal{L}(\frac{\sigma}{\sigma_{nl}}T_{nl}|Z_{nl-1}),\mathcal{L}(B)) && (6.54)\\
&\leq c_E[\frac{c(r,\xi)}{(LZ_{nl-1})^r}]^{1/4}I(Z_{nl-1}>0) && (6.55)
\end{aligned}$$

on $B_{n\alpha}$. Of course, $\mathcal{L}(B)$ is the law of the standard Brownian motion on $C_0[0,1]$. Now eventually on S_0 we have $Z_n > \beta^n$ for $1 < \beta < m$ and hence almost surely on S_0 eventually we have that the Prokhorov distance b_n is less than $1/n$. Thus $\lim_{n\to\infty} b_n(L(nl))^{1/2} = 0$ there, and almost surely on S_0 we have eventually in n that

$$P(M_B \in [(c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,\frac{5}{2}\alpha)]^{2b_n}) \leq P(M_B \in (c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,3\alpha)),$$

where the probability inequality follows from simple set inclusion. Hence on S_0 we have eventually in n that

$$\theta_{n,1} \leq \psi_{n,1} + \psi_{n,2} + II_n, \tag{6.56}$$

where

$$\psi_{n,1} = E(I(G_n|\mathcal{H}_{n-1})P(M_B \in (c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,3\alpha)) \tag{6.57}$$

and

$$\psi_{n,2} = c_Ec(r,\xi)^{1/4}E(I(G_n \cap B_{n,\alpha})(LZ_{nl-1})^{-r/4}|\mathcal{H}_{n-1}). \tag{6.58}$$

Now,

$$\sum_{n\geq 1} E(\psi_{n,2}) \leq \sum_{n\geq 1} E((LZ_{nl-1})^{-r/4} I(Z_{nl-1>0})) < \infty \tag{6.59}$$

by the harmonic moment results of Appendix A and that $r > 4$. Hence $\sum_{n\geq 1} \psi_{n,2}$ converges with probability one and on S_0 we have

$$P(E_{nl}|\mathcal{H}_{n-1}) \leq \psi_{n,1} + \psi_{n,3}, \tag{6.60}$$

where

$$\psi_{n,3} = \theta_{n,2} + \psi_{n,2} + II_n \tag{6.61}$$

and $\theta_{n,2}$, $\psi_{n,2}$, and II_n are summable with probability one. Furthermore, since the term

$$P(M_B \in (c_2/L(nl))^{1/2} N(f_1, \mathcal{P}_1, 3\alpha))$$

in (6.57) is deterministic, we have $\sum_{n\geq 1} \psi_{n,1} < \infty$ almost surely on S_0 if

$$\sum_{n\geq 1} E(I(G_n|\mathcal{H}_{n-1}) P(M_B \in (c_2/L(nl))^{1/2} N(f_1, \mathcal{P}_1, 3\alpha)) < \infty$$

almost surely on S_0. Hence we need to study this last series, and recalling that G_n involves one less of the sets $A_{nl,j,\alpha}$, we iterate the above argument $l-1$ more times, starting at (6.30-33) with subsequent analogues of $B_{n,\alpha}$, to obtain on S_0 for all sufficiently large n that

$$P(E_{nl}|\mathcal{H}_{n-1}) \leq \{\prod_{j=1}^{l} P(M_B \in (c_2/L(nl))^{1/2} N(f_j, \mathcal{P}_j, 3\alpha) I(Z_{(n-1)l} > \beta^{(n-1)l})\} + \psi_{n,4} \tag{6.62}$$

where $\sum_{n\geq 1} \psi_{n,4} < \infty$. Now we apply Proposition 2.2 of [6] for Wiener measure, which holds even if the $a_j's$ and $b_j's$ mentioned there merely satisfy $a_j < b_j$ for $1 \leq j \leq m$ and $b_1 \leq b_2 \leq, \cdots \leq b_m$. Thus our choice of α in forming the open set V as in (6.19) implies for $\gamma > 0$ and for all sufficiently large n that

$$P(E_{nl}|\mathcal{H}_{n-1}) \leq \exp\{-\log_e(nl)(1-\gamma)(1+\delta)\} I(Z_{(n-1)l} > \beta^{(n-1)l}). \tag{6.63}$$

Now taking γ sufficiently small so that $(1-\gamma)(1+\delta) > 1$, we have a.s. on S_0 that

$$\sum_{n\geq 1} P(E_{nl}|\mathcal{H}_{n-1}) < \infty. \tag{6.64}$$

The proof of the lemma now follows as indicated from (6.27), since the other $l-1$ cases are completely similar. ∎

Hence from this last lemma we have almost surely on the survival set S that

$$C(\{(Ln/c_2)^{1/2} \mathbf{M}_{n,r(n)}\}) \subseteq \mathcal{K}_\infty,$$

when we use the product topology on $(C_0[0,1])^\infty$. Our next lemma establishes that in this setting the cluster set $C(\{(Ln/c_2)^{1/2} \mathbf{M}_{n,r(n)}\})$ is actually $\mathcal{K}_\infty$ almost surely on S.

Lemma 19. *Let S denote the survival set of the process and set $c_2 = \pi^2/8$. Then, under the conditions of the theorem we have*

$$P(\{C(\{(Ln/c_2)^{1/2}\mathbf{M}_{n,r(n)}\}) = \mathcal{K}_\infty\} \cap S) = P(S). \tag{6.65}$$

Proof. Since the cluster set of a sequence of points in $((C_0[0,1])^\infty, \rho_\infty)$ is closed, and the topological space $((C_0[0,1])^\infty, \rho_\infty)$ is separable, it is sufficient to show that for an arbitrary point $\mathbf{f} \in \mathcal{K}_\infty$ with $\sum_{j\geq 1}\int_0^1 f_j^{-2}(s)ds \leq 1$, we have a.s. on S_0 that

$$(Ln/c_2)^{1/2}\mathbf{M}_{n,r(n)} \in V \quad i.o. \tag{6.66}$$

where V is an arbitrarily small open set containing $\mathbf{f}$. Furthermore, if

$$\mathcal{K}_0 = \{\mathbf{f} = (f_1, f_2, \cdots) \in \mathcal{K}_\infty : \ f_j(1) < \infty \text{ for all } j \geq 1, \sum_{j\geq 1}\int_0^1 f_j^{-2}(s)ds < 1\},$$

then by Lemma 16 we see $\mathcal{K}_0$ is dense in $\mathcal{K}_\infty$. Hence it suffices to show that almost surely on S_0 (6.66) holds for each $\mathbf{f} \in \mathcal{K}_0$, when $V = \prod_{j=1}^\infty V_j$ is an open set containing $\mathbf{f}$ of the form in Lemma 18. That is, since $f_j(1) < \infty$ for all $j \geq 1$ when $\mathbf{f} = (f, f_2, \cdots) \in \mathcal{K}_0$, then by Lemma 15 it suffices to take l an arbitrary positive integer, partitions $\mathcal{P}_j$ of continuity points of f_j for $1 \leq j \leq l$, and a single $\alpha > 0$ arbitrarily small with

$$V_j = N(f_j, \mathcal{P}_j, \alpha), \quad 1 \leq j \leq l,$$

and $V_j \in \mathcal{M}$ for $j \geq l+1$. Moreover, by replacing $\mathbf{f} = (f_1, f_2, \cdots) \in \mathcal{K}_0$ by $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \cdots)$ where

$$\tilde{f}_j(t) = f_j(t) + \frac{\epsilon t}{2^j}, \quad 0 \leq t \leq 1,$$

and $\epsilon > 0$ is arbitrarily small, there is no loss in generality in assuming that each f_j is strictly increasing on $[0,1]$. Thus we also assume this holds for our $\mathbf{f} \in \mathcal{K}_0$.

By the conditional Borel Cantelli lemma it suffices to show that

$$\sum_{n\geq 1} P(E_{nl}|\mathcal{H}_{n-1}) = \infty \tag{6.67}$$

where E_{nl} and $\mathcal{H}_{n-1}$ are as before in Lemma 18, except now $\mathbf{f} = (f_1, f_2, \cdots) \in \mathcal{K}_0$ so we also have $\sum_{j\geq 1}\int_0^1 f_j^{-2}(s)ds < 1$, and l and α are arbitrary but fixed in our argument. Thus to verify (6.67), observe that for all n sufficiently large, on S_0

$$\begin{aligned} P(E_{nl}|\mathcal{H}_{n-1}) &= E(E(I(A_{nl,0,\alpha}|\mathcal{F}_{nl-1})I(G_n)|\mathcal{H}_{n-1}) & (6.68)\\ &> \theta_{n,1} - \theta_{n,2}, & (6.69) \end{aligned}$$

where

$$\theta_{n,1} = E(I(G_n)E(I(A_{nl,0,\alpha} \cap B_{n,\alpha}|\mathcal{F}_{nl-1})|\mathcal{H}_{n-1}), \tag{6.70}$$

$$\theta_{n,2} \leq P(B'_{n,\alpha}|\mathcal{H}_{n-1}), \tag{6.71}$$

and

$$B_{n,\alpha} = \{c_E[c(r,\xi)(LZ_{nl-1})^{-r}]^{1/4} < \rho_0, Z_{nl-1} > r_0(f_1, \cdots f_l; \alpha) > 1\}, \tag{6.72}$$

where c_E and $c(r,\xi)$ are defined as above. Of course, here G_n is as in (6.50) with the sets $A_{nl,j,\alpha}$ defined as before except that now they are defined in terms of the sets $N(f_j,\mathcal{P}_j,\alpha)$. Also, here we take $r_0(f_1,\cdots f_l,\alpha)$ such that $Z_k > r_0(f_1,\cdots f_l,\alpha)$ implies

$$\frac{\sigma}{\sigma_k}N(f_j,\mathcal{P}_j,\frac{3}{4}\alpha) \supseteq N(f_j,\mathcal{P}_j,\frac{\alpha}{2}) \tag{6.73}$$

for $j=1,2,\cdots,l$. From (5.90) and (5.91), $\sum_{n\geq 1}\theta_{n,2}<\infty$ a.s. on Ω, and arguing as in (5.92-95) and recalling that $V_1 = N(f_1,\mathcal{P}_1,\alpha)$,

$$\begin{aligned} P(A_{nl,0,\alpha}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}) &= P(\{\eta_{nl,Z_{nl-1}}\in V_1 = N(f_1,\mathcal{P}_1,\alpha)\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}) && (6.74)\\ &\geq I_n - II_n, && (6.75)\end{aligned}$$

where

$$I_n = P(\{(L(nl)/c_2)^{1/2}\Lambda(T_{nl})\in N(f_1,\mathcal{P}_1,\frac{3}{4}\alpha\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}), \tag{6.76}$$

and

$$II_n = P(\{||\eta_{nl,Z_{nl-1}} - (L(nl)/c_2)^{1/2}\Lambda(T_{nl})|| > \frac{\alpha}{4}\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}). \tag{6.77}$$

Hence,

$$\begin{aligned} \theta_{n,1} &\geq E(I(G_n)(I_n - II_n)|\mathcal{H}_{n-1}) && (6.78)\\ &\geq E(I(G_n)I_n|\mathcal{H}_{n-1}) - E(II_nI(Z_{(n-1)l}>\beta^{(n-1)l})|\mathcal{H}_{n-1}), && (6.79)\end{aligned}$$

where as in (6.45-50), we have $\sum_{n\geq 1}E(II_nI(Z_{(n-1)l}>\beta^{(n-1)l})|\mathcal{H}_{n-1})<\infty$ with probability one.

Recalling the definition of $B_{n,\alpha}$ in this setting, we see that $Z_{nl-1} > r_0(f_1,f_2,\cdots f_l;\alpha)$ implies

$$\frac{\sigma}{\sigma_k}N(f_j,\mathcal{P}_j,\frac{3}{4}\alpha) \supseteq N(f_j,\mathcal{P}_j,\frac{\alpha}{2}),$$

and we also have that

$$P(M_B\in(c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,\frac{\alpha}{4}))I(B_{n,\alpha})$$
$$\leq P(\{\Lambda(\frac{\sigma}{\sigma_{nl}}T_{nl})\in[(c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,\frac{\alpha}{4})]^{2b_n}\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}) + 2b_nI(B_{n,\alpha}),$$

where b_n denotes the Prokorov distance

$$\rho(\mathcal{L}(\frac{\sigma}{\sigma_{nl}}T_{nl}|Z_{nl-1}>0),\mathcal{L}(B)).$$

Furthermore, Corollary 2 of Einmahl, ([12]), and Lemma 8 implies we have on $B_{n,\alpha}$ that

$$b_n \leq c_E[c(r,\xi)(LZ_{nl-1})^{-r}]^{1/4}\leq \rho_0.$$

In addition, $\lim_{n\to\infty} b_n(L(nl))^{\frac{1}{2}}$ on S_0 and $b_n<\rho_0$ on $B_{n,\alpha}$. Hence if $\rho_0<\alpha/12$ the above implies for all n sufficiently large that

$$P(M_B\in(c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,\frac{\alpha}{4}))I(B_{n,\alpha})$$

$$\leq P(\{\Lambda(\frac{\sigma}{\sigma_{nl}}T_{nl})\in[(c_2/L(nl))^{1/2}N(f_1,\mathcal{P}_1,\frac{\alpha}{4})]^{2b_n}\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}) + 2b_nI(B_{n,\alpha})$$

$$\leq P(\{\Lambda(\frac{\sigma}{\sigma_{nl}}T_{nl}) \in (c_2/L(nl))^{1/2}N(f_1, \mathcal{P}_1, \frac{\alpha}{2})\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}) + 2b_nI(B_{n,\alpha})$$

$$\leq P(\{\Lambda(T_{nl}) \in (c_2/L(nl))^{1/2}N(f_1, \mathcal{P}_1, \frac{3}{4}\alpha)\} \cap B_{n,\alpha}|\mathcal{F}_{nl-1}) + 2b_nI(B_{n,\alpha}),$$

where Lemma 17 is used in the first inequality and M_B is as defined in (6.10). Now $\sum_{n\geq 1} E(b_nI(B_{n,\alpha})|\mathcal{H}_{n-1}) < \infty$ a.s. on Ω by the harmonic moment results of Appendix A ; i.e. $b_n < \rho_0$ implies $Z_{nl-1} > 0$. In addition, we also have $\sum_{n\geq 1} \theta_{n,2} < \infty$ a.s. on Ω. Thus the above shows it suffices to verify

$$\sum_{n\geq 1} P(G_n \cap B_{n,\alpha}|\mathcal{H}_{n-1})P(M_B \in (c_2/L(nl))^{1/2}N(f_1, \mathcal{P}_1, \frac{\alpha}{4})) = \infty \tag{6.80}$$

a.s. on S_0. Now,

$$P(G_n \cap B_{n,\alpha}|\mathcal{H}_{n-1}) = P(G_n|\mathcal{H}_{n-1}) - P(B'_{n,\alpha}|\mathcal{H}_{n-1}) \tag{6.81}$$

and since $\sum_{n\geq 1} P(B'_{n,\alpha}|\mathcal{H}_{n-1}) < \infty$ with probability one by what we did earlier, (6.80) will follow if we show that a.s. on S_0

$$\sum_{n\geq 1} P(G_n|\mathcal{H}_{n-1})P(M_B \in (c_2/L(nl))^{1/2}N(f_1, \mathcal{P}_1, \frac{\alpha}{4})) = \infty. \tag{6.82}$$

Iterating the previous argument $l-1$ more times we see as before that since the quantities

$$P(M_B \in (c_2/L(nl))^{1/2}N(f_j, \mathcal{P}_j, \frac{\alpha}{4}))$$

are deterministic, it suffices to show that a.s. on S_0

$$\sum_{n\geq 1}\prod_{j=1}^{l} P(M_B \in (c_2/L(nl))^{1/2}N(f_j, \mathcal{P}_j, \frac{\alpha}{4}))I(Z_{(n-1)l} > \beta^{(n-1)l}) = \infty. \tag{6.83}$$

Since

$$P(\{Z_{(n-1)l} > \beta^{(n-1)l} \ eventually\} \cap S_0) = P(S_0), \tag{6.84}$$

it is therefore sufficient to show that

$$\sum_{n\geq 1}\prod_{j=1}^{l} P(M_B \in (c_2/L(nl))^{1/2}N(f_j, \mathcal{P}_j, \frac{\alpha}{4})) = \infty. \tag{6.85}$$

Now we apply Proposition 2.4 of ([6]) for Brownian motion, which implies for $\gamma > 0$ and $j = 1, \cdots, l$ that

$$P(M_B \in (c_2/L(nl))^{1/2}N(f_j, \mathcal{P}_j, \frac{\alpha}{4})) \geq \exp\{-(1+\gamma)L(nl \sum_{t_k\in\mathcal{P}_j} (t_k - t_{k-1})/(f_j(t_k) + \frac{\alpha}{4})^2\} \tag{6.86}$$

provided $n \geq n(\gamma)$. Since the $f_j's$ are increasing we have

$$\sum_{j=1}^{l}\{\sum_{t_k\in\mathcal{P}_j} (t_k - t_{k-1})/(f_j(t_k) + \frac{\alpha}{4})^2\} \leq \sum_{j=1}^{l}\int_0^1 f_j^{-2}(s)ds,$$

and since

$$\sum_{j\geq 1}\int_0^1 f_j^{-2}(s)ds < 1,$$

there exists $\gamma > 0$ sufficiently small so that for $n \geq \tilde{n}(\gamma)$ we have

$$\prod_{j=1}^{l} P(M_B \in (c_2/L(nl))^{1/2} N(f_j, \mathcal{P}_j, \frac{\alpha}{4})) \geq \exp\{-Ln\}. \tag{6.87}$$

Thus we can choose $\gamma > 0$ sufficiently small so that for all n sufficiently large

$$\prod_{j=1}^{l} P(M_B \in (c_2/L(nl))^{1/2} N(f_j, \mathcal{P}_j, \frac{\alpha}{4})) \geq \frac{1}{n} \tag{6.88}$$

yielding (6.85). This proves (6.67) and the lemma. ∎

Replacing the normalizer Z_{n-1} in the denominator of the definition of $X_{n,Z_{n-1}}(t)$ in (2.12) by m^{n-1} suggests the possibility that a result similar to Proposition 2 might hold for our analogue of the Chung-Wichura theorem. For these purposes we define the processes on $[0,1]$ given by

$$M_{n,Z_{n-1},m^{n-1}}(t) = (\frac{Z_{n-1}}{m^{n-1}})^{\frac{1}{2}} M_{n,Z_{n-1}}(t), \quad 0 \leq t \leq 1, \tag{6.89}$$

and

$$\mathbf{M}_{n,r(n),m^{n-1}}(t) = (M_{n,Z_{n-1},m^{n-1}}(t), \cdots, M_{n-r(n)+1,Z_{n-r(n)},m^{n-r(n)}}(t), 0, 0. \ldots). \tag{6.90}$$

Then the following holds.

Proposition 3. *Assume $E(Z_1^2(L(Z_1))^r) < \infty$ for some $r > 4$, that $1 \leq r(n) \leq n$, and $\lim_{n\to\infty} r(n) = \infty$. Let W be given as in Proposition 1, and S the survival set of the process $\{Z_n : n \geq 0\}$. Then*

$$P(\lim_{n\to\infty} \rho_\infty(\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n),m^{n-1}}, W^{\frac{1}{2}}\mathcal{K}_\infty) = 0 | S) = 1, \tag{6.91}$$

where the ρ_∞-distance from a point to a set is defined as usual. In addition, if clustering is determined with respect to the product topology on $\mathcal{M}^\infty$, then we have

$$P(C(\{\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n),m^{n-1}}\}) = W^{\frac{1}{2}}\mathcal{K}_\infty | S) = 1. \tag{6.92}$$

Proof. First of all observe that

$$\mathbf{M}_{n,r(n),m^{n-1}}(t) = ((\frac{Z_{n-1}}{m^{n-1}})^{\frac{1}{2}} M_{n,Z_{n-1})}(t), \cdots, (\frac{Z_{n-r(n)}}{m^{n-r(n)}})^{\frac{1}{2}} M_{n-r(n)+1,Z_{n-r(n)}}(t), 0, 0, \ldots), \tag{6.93}$$

and Theorem 5 implies almost everywhere on S that

$$\lim_{n\to\infty} \rho_\infty(\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n)}, \mathcal{K}_\infty) = 0. \tag{6.94}$$

Letting $E = \{\omega \in S : \lim_{n\to\infty} \frac{Z_{n-1}}{m^{n-1}}(\omega) = W(\omega) < \infty\}$, we have $P(E \triangle S) = 0$. Now fix $\omega \in E$ such that (6.94) holds. Then the relative compactness of $\mathcal{M}^\infty$, or that $\mathcal{K}_\infty$ is compact and (6.94) holds, implies for every subsequence n' of the integers there is a further subsequence n'', possibly depending on ω, and $\mathbf{k}(\cdot,\omega) = (k_1(\cdot,\omega), k_1(\cdot,\omega), k_1(\cdot,\omega), \cdots) \in \mathcal{K}_\infty$ such that

$$\lim_{n''\to\infty} \rho_\infty(\sqrt{\frac{Ln''}{c_2}}\mathbf{M}_{n'',r(n'')}(\cdot,\omega), \mathbf{k}(\cdot,\omega)) = 0, \tag{6.95}$$

which implies

$$\lim_{n''\to\infty}\rho(\sqrt{\frac{Ln''}{c_2}}M_{n''-j+1,Z_{n''-j}}(\cdot,\omega),k_j(\cdot,\omega))=0 \tag{6.96}$$

for all $j\geq 1$. Furthermore, since $\omega\in E$, we also have

$$\lim_{n\to\infty}\frac{Z_{n-1}(\omega)}{m^{n-1}}=W(\omega)<\infty. \tag{6.97}$$

Now

$$\lim_{n\to\infty}\rho_\infty(\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n),m^{n-1}}(\cdot,\omega),W^{\frac{1}{2}}(\omega)\mathcal{K}_\infty)=0 \tag{6.98}$$

if and only if for each subsequence n' of the integers, there is a further subsequence n'', possibly depending on ω, and $\mathbf{h}(\cdot,\omega)\in\mathcal{K}_\infty$ such that

$$\lim_{n''\to\infty}\rho_\infty(\sqrt{\frac{Ln''}{c_2}}\mathbf{M}_{n'',r(n''),m^{n''-1}}(\cdot,\omega),W^{\frac{1}{2}}(\omega)\mathbf{h}(\cdot,\omega)_\infty)=0. \tag{6.99}$$

Furthermore, (6.99) holds if and only if for all $j\geq 1$

$$\lim_{n''\to\infty}\rho(\sqrt{\frac{Ln''}{c_2}}M_{n''-j+1,Z_{n''-j},m^{n''-j}}(\cdot,\omega),W^{\frac{1}{2}}(\omega)h_j(\cdot,\omega))=0. \tag{6.100}$$

Hence for $\omega\in E$ such that (6.94) holds, take a subsequence n' of the integers. Now take a further subsequence n'' and $\mathbf{k}(\cdot,\omega)\in\mathcal{K}_\infty$ such that (6.95) holds . Also let $\mathbf{h}(\cdot,\omega)=\mathbf{k}(\cdot,\omega)$, where $\mathbf{k}(\cdot,\omega)$ is as in (6.95). Now (6.100) holds for each $j\geq 1$ if for each $t\in[0,1]$ which is a continuity point of $k_j(\cdot,\omega)$ we have

$$\lim_{n''\to\infty}\sqrt{\frac{Ln''}{c_2}}M_{n''-j+1,Z_{n''-j},m^{n''-j}}(t,\omega)=W^{\frac{1}{2}}(\omega)k_j(t,\omega). \tag{6.101}$$

However,

$$\sqrt{\frac{Ln}{c_2}}M_{n-j+1,Z_{n-j},m^{n-j}}(t,\omega)=(\frac{Z_{n-1}}{m^{n-1}})^{\frac{1}{2}}\sqrt{\frac{Ln}{c_2}}M_{n-j+1,Z_{n-j}}(t,\omega), \tag{6.102}$$

and hence whenever $\omega\in E$ and (6.96) holds for all $j\geq 1$, we have (6.101) holding for all $j\geq 1$. Thus (6.91) is proven.

To finish the proof of Proposition 3 one proves (6.92) by repeating the argument above arguing along suitable subsequences. That is, if $\mathbf{k}$ is in the cluster set

$$C(\{\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n)}\}), \tag{6.103}$$

and $\omega\in E$, then $W^{\frac{1}{2}}(\omega)\mathbf{k}$ is in the cluster set

$$C(\{\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n),m^{n-1}}\}). \tag{6.104}$$

Hence almost surely on S we have $C(\{\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n),m^{n-1}}\})=W^{\frac{1}{2}}\mathcal{K}_\infty$, and (6.92) holds. ∎

7 Appendix A

In this section we will establish the rate of convergence of the harmonic moment of $(LZ_n)^r$. Recall that $Lt = \max\{1, \log_e t)\}$ for $t \geq 0$. The method of investigation involves the integral representation of $(LZ_n)^r$ used in the study of the harmonic moments of Z_n^r [26]. Useful facts to aid in the proof of the harmonic moments of LZ_n result are contained in the following two lemmas.

Lemma 20. *Let $0 < b < 1/2$ and suppose $r > 0$.Then there exists $c(r) < \infty$ such that*

$$I(r,b) = \int_1^\infty b^x x^{r-1} dx \leq c(r) b / \log_e(1/b). \tag{7.1}$$

Proof. When $0 < r \leq 1$, then x^{r-1} is decreasing in x and hence

$$I(r,b) \leq \int_1^\infty b^x dx = b / \log_e(1/b),$$

so the lemma holds when $0 < r \leq 1$ with $c(r) = 1$. When $r > 1$, by integrating by parts, it follows that

$$I(r,b) = \frac{b}{\log_e(\frac{1}{b})} + \frac{(r-1)}{\log_e(\frac{1}{b})} I(r-1,b). \tag{7.2}$$

If $0 < r - 1 \leq 1$, then x^{r-2} is decreasing in x, and as in the previous case we see

$$I(r-1,b) \leq b / \log_e(1/b).$$

Thus by combining the previous two inequalities we again see the result holds with $c(r) = 1 + (r - 1)/\log_e 2$ for $1 < r \leq 2$. If $r - 1 > 2$, by iterating the above, the lemma follows. ■

Lemma 21. *Let $0 < b < 1$ and suppose $r > 1$. Furthermore, let $0 < \tau < 1$ and define*

$$J_1(r,b) = \int_0^\tau b^x x^{r-1} dx,$$

and

$$J_2(r,b) = \int_\tau^1 b^x x^{r-1} dx.$$

Then $J_1(r,b)$ and $J_2(r,b)$ are increasing in b for $0 < b < 1$,

$$J_1(r,b) \leq \tau^r / r, \tag{7.3}$$

and

$$J_2(r,b) \leq b^\tau / \log_e(1/b). \tag{7.4}$$

Proof. That these functions increase in b is obvious. Furthermore, since $b^x \leq 1$ for $x > 0$, we have

$$J_1(b,r) \leq \int_0^\tau x^{r-1} dx = \tau^r / r,$$

and hence the estimate for $J_1(r,b)$ is immediate. Now $r \geq 1$ implies $x^{r-1} \leq 1$ on $[\tau, 1]$, and hence

$$J_2(r,b) \leq \int_\tau^1 b^x dx \leq b^\tau / \log_e(1/b).$$

Thus the lemma is proven. ■

Theorem 7. *Let $r > 0$, and assume $\{Z_n : n \geq 0\}$ is a Galton-Watson branching process with $1 < m = E(Z_1) < \infty$. Then*

$$\limsup_{n\to\infty} n^r E((LZ_n)^{-r} I(Z_n > 0))/(\log_e n)^r < \infty. \tag{7.5}$$

Proof. Let $\beta(n)$ be an arbitrary sequence of real numbers such that $\lim_{n\to\infty} \beta(n) = \infty$. Then we will prove that

$$\limsup_{n\to\infty} n^r E((LZ_n)^{-r} I(Z_n > 0))/(\beta(n)\log_e n)^r = 0. \tag{7.6}$$

Of course, since the sequence $\{\beta(n)\}$ converges to infinity as slowly as we wish, this suffices to prove (7.5). Hence it remains to verify (7.6).

Since $Lt = 1$ for $0 \leq t \leq e$ we first observe that

$$E((LZ_n)^{-r} I(Z_n > 0)) = P(1 \leq Z_n \leq 2) + E((LZ_n)^{-r} I(Z_n \geq 3)).$$

Using Lemma 4 it follows that

$$\lim_{n\to\infty} n^r P(1 \leq Z_n \leq 2) = 0,$$

and hence it suffices to show

$$\lim_{n\to\infty} n^r E((LZ_n)^{-r} I(Z_n \geq 3))/(\beta(n)\log_e n)^r = 0.$$

Now the definition of the gamma function implies we have for every $y > 0$ that

$$\frac{1}{y^r} = \frac{1}{\Gamma(r)} \int_0^\infty \exp(-\theta y)\theta^{r-1}\, d\theta,$$

and hence by replacing y by (LZ_n), we get

$$\begin{aligned} (LZ_n)^{-r} I(Z_n \geq 3) &= \frac{1}{\Gamma(r)} \int_0^\infty \exp(-\theta(LZ_n))\theta^{r-1}\, d\theta I(Z_n \geq 3) &(7.7)\\ &= \frac{1}{\Gamma(r)} \int_0^\infty Z_n^{-\theta}\theta^{r-1}\, d\theta I(Z_n \geq 3). &(7.8) \end{aligned}$$

Then Tonelli's Theorem implies

$$\begin{aligned} E((LZ_n)^{-r} I(Z_n \geq 3)) &= \frac{1}{\Gamma(r)} \int_0^\infty E(Z_n^{-\theta} I(Z_n \geq 3))\theta^{r-1}\, d\theta &(7.9)\\ &= A_n + B_n, &(7.10) \end{aligned}$$

where Jensen's inequality implies

$$A_n \leq \frac{1}{\Gamma(r)} \int_0^1 [E(Z_n^{-1} I(Z_n \geq 3))]^\theta \theta^{r-1}\, d\theta,$$

and

$$B_n = \frac{1}{\Gamma(r)} \int_1^\infty E(Z_n^{-\theta} I(Z_n \geq 3))\theta^{r-1}\, d\theta.$$

Our next concern is to show $\lim_{n\to\infty} n^r B_n = 0$. To check this observe that

$$B_n = \frac{1}{\Gamma(r)} \int_1^\infty \sum_{j\geq 3} \{ \int_1^\infty j^{-x} P(Z_n = j) x^{r-1}\, dx,$$

and by Tonelli's Theorem, and applying (7.1) of Lemma 14 with $b = j^{-1}$, we see that

$$B_n = \frac{1}{\Gamma(r)} \sum_{j \geq 3} c(r)(j \log_e j)^{-1} P(Z_n = j) \leq \frac{c(r)}{\Gamma(r)} E(Z_n^{-1} I(Z_n \geq 1)).$$

Thus by applying (21),(23), and (25) in the proof of (4) in Theorem 2 of [19], there exists $0 < \gamma_{HB} < 1$ such that

$$E(Z_n^{-1} I(Z_n \geq 3)) \leq \gamma_{HB}^n, \tag{7.11}$$

and hence we have

$$\lim_{n\to\infty} n^r B_n \leq \frac{c(r)}{\Gamma(r)} \lim_{n\to\infty} n^r \gamma_{HB}^n = 0.$$

To finish the proof, it thus suffice to show

$$\lim_{n\to\infty} n^r A_n / (\beta(n) \log_e n)^r = 0.$$

Again, by applying (4) in Theorem 2 of [19] we have (7.11), and hence for $0 < \tau < 1$

$$\Gamma(r) A_n \leq J_1(\tau, \gamma_{HB}^n) + J_2(\tau, \gamma_{HB}^n)).$$

Taking $\beta(n) \to \infty$ slowly enough that $0 < \tau \equiv \beta^{\frac{1}{2}}(n)(\log_e n)/n < 1$ we see from Lemma 15 that

$$\Gamma(r) A_n \leq \frac{1}{r} (\beta^{\frac{1}{2}}(n)(\log_e n)/n)^r + (\gamma_{HB}^n)^{\beta^{\frac{1}{2}}(n)(\log_e n)/n} / \log_e(\gamma_{HB}^n).$$

Thus $\lim_{n\to\infty} n^r A_n / (\beta(n) \log_e n)^r = 0$ as required, and the theorem is proved. ∎

8 Appendix B

The table below reports the average estimate of the change point and the fraction of iterations where the estimate was correct, i.e. where the estimate equalled $n^\star$. Recall, from Remark 3-Remark 6 in Section 2, that k denotes the number of consecutive confidence intervals used to identify the change point.

Case 1, $n^\star = 11$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
fraction correct	0.162	0.587	0.866	0.952	0.987	0.996	0.998	0.998
average	3.145	7.248	9.778	10.556	10.873	10.96	10.996	10.996

Case 2, $n^\star = 16$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
fraction correct	0.014	0.158	0.431	0.638	0.775	0.888	0.943	0.965	0.976	0.981
average	1.723	4.199	8.147	11.036	12.907	14.468	15.242	15.57	15.764	15.85

Case 3, $n^\star = 25$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
fraction correct	0.519	0.926	0.991	0.999	1
average	13.781	23.272	24.787	24.976	25

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