

# Paradoxes of Voting

Alexander Tabarrok

Department of Economics

George Mason University

Tabarrok@gmu.edu\*

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\*The first two characters in the email address are zeroes.

## 1. Introduction

Figure 1.1 shows one way of organizing our conceptual apparatus. A voting system takes preferences as input and outputs an outcome. In the previous chapter, we showed that if we input the same preferences into different voting systems we can get very different outputs. In this chapter we show that holding the voting system constant, changes in preferences can lead to paradoxical changes in outcomes. Of course a change in preferences *should* lead to a changes in outcomes - democracy wouldn't mean much if the voting system always outputted the same ranking regardless of inputs - what we are going to show is that when preferences change, the outcome can change in ways which are unexpected and often undesirable.

Since this chapter is rather long and full of excursions we provide a roadmap. We are first going to demonstrate the paradox of *Intransitivity of Group Preferences*. We then will discuss at some length various implications of this paradox such as the possibility of making Pareto dominated choices, agenda setting, and killer amendments. Our second paradox is the *Failure of Positive Association*. The third and fourth paradoxes are two forms of the *Failure of Independence of Irrelevant Alternatives*. The first version we call the *Dropping out Paradox* the

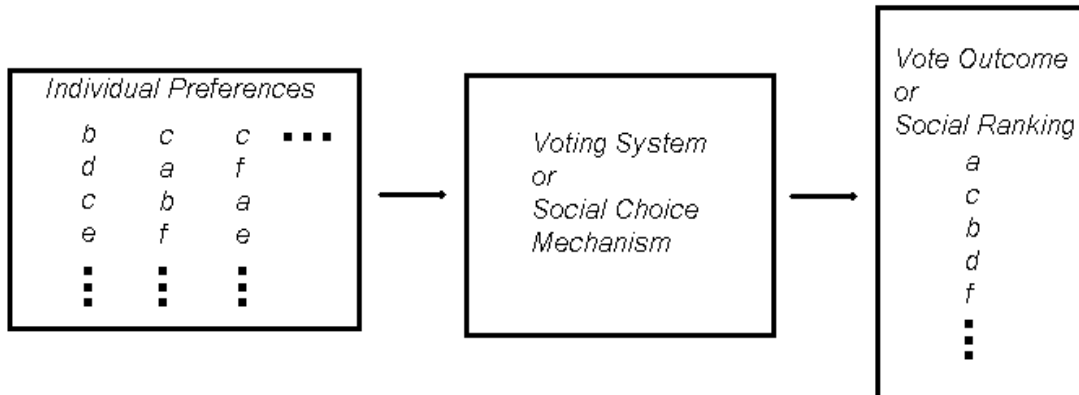


Figure 1.1: A voting system or social choice mechanism aggregates individual preference orderings into a social preference ordering.

second the *Changing Preferences Paradox*.

## 2. Intransitivity of Group Preferences

### 2.1. Pairwise comparisons using majority rule can lead to cycling

Imagine how odd it would seem if someone told you that they preferred Apples to Bananas and Bananas to Coconuts but Coconuts to Apples. If an individual had these preferences an economist would call him or her irrational because these preferences violate the *transitivity* axiom. If we let  $\succ$  mean ‘is preferred to’ and  $\succeq$  mean ‘is preferred or indifferent to’ then the transitivity axiom says that if  $A \succ B$  and  $B \succeq C$  then  $A \succeq C$  so we can write  $A \succeq B \succeq C$ . Economists

demand that any representation of a person's preferences obey the transitivity axiom because a person with intransitive preferences can be made to act against their own self-interest. Suppose that Joe has the preferences Apples  $\succ$  Bananas, Bananas  $\succ$  Coconuts and Coconuts  $\succ$  Apples and imagine that we own a coconut, a banana, and an apple. Let's sell the coconut to Joe for \$1. Joe prefers bananas to coconuts so he will willingly give us back the coconut and a little bit of money, say ten cents, in return for the Banana. Joe also prefers apples to bananas so he will willingly give us back the banana and a little bit of money, say 10 cents, in return for the apple. Since Joe likes coconuts more than apples he willingly give us back the apple plus say 10 cents in return for the coconut. But we are now back where we began! Except, Joe is 30 cents poorer and we are 30 cents richer. By repeating the process we can use Joe as a *money pump* and take from him all of his wealth. Non-transitive preferences can be very costly!

The most famous voting paradox was first discovered by the Marquis de Condorcet (1785). Condorcet discovered that a group which uses majority rule to make decisions can behave as if its 'preferences' are intransitive even if every individual in the group has transitive preferences. Speaking loosely, Condorcet showed that groups can have irrational preferences. If there are only two issues then majority rule works just fine but say we must choose between three or more

issues. One method of making this choice is by pairwise comparisons each using majority rule. If our options are A, B and C then a good rule for choice might be to choose that option which can beat any other option in a majority rule contest. Condorcet showed that such an option might not exist. Or, as we know say, majority rule can fail to produce a Condorcet winner. The following example illustrates the paradox. Consider three voters with preferences as given by Table One.

Table One: Cyclic Preferences			
	Voter One	Voter Two	Voter Three
First Choice	<i>A</i>	<i>C</i>	<i>B</i>
2nd	<i>B</i>	<i>A</i>	<i>C</i>
3rd	<i>C</i>	<i>B</i>	<i>A</i>

Consider a vote of  $A \succ B$ : Voter One prefers  $A$  to  $B$  and so votes for  $A$ , Voter Two also votes for  $A$  (since  $A$  is ranked above  $B$ ), and Voter Three votes for  $B$ . By majority rule  $A$  beats  $B$ . Similarly  $C$  beats  $A$ . If we use  $\succ$  to indicate “beats” or “is preferred to” it seems reasonable to believe that if  $C \succ A$  and  $A \succ B$  then  $C \succ B$ . But consider voting  $C \succ B$ : Voter One votes for  $B$ , Voter Two votes for  $C$  and Voter Three votes for  $B$  so by majority rule  $B \succ C$ !

This paradox is often called the voting cycle paradox because, given the above

preferences, the search for a choice which beats all others goes on forever. Voters continually switch between  $A$ ,  $B$ , and  $C$  so long as voting continues. An outsider could money pump this group forever or at least until it ran out of money or changed to a different voting system! Vote cycling does not always occur. If voter three, for example, had had the preference  $C \succ B \succ A$  then  $C$  would have been a Condorcet winner, ie.  $C$  would have beaten all other choices in pairwise voting. We will discuss later whether preferences which give rise to cycling are likely or unlikely.

The fact that majority rule can lead to behavior which we would consider irrational if displayed by an individual has led some to suggest that society's preferences can be irrational or the "will of majority" can be incoherent. If you believe that society has preferences, as many people implicitly do, these statements are true. On the other hand one might ask, Why should we expect society to have rational preferences when the concept of society having preferences is meaningless to begin with (Buchanan, 1954)? Society does not have preferences any more than society has bad breath. Why should we care if something which does not exist (society's preferences) does or does not display certain properties (transitivity)? This view is further supported by noting that we are not much bothered by intransitivities in other areas of life. For example, suppose there are three contenders

for the world heavyweight boxing champion, Mike Tyson, Muhammad Ali, and Rocky Marciano. It wouldn't be impossible or even unusual if Tyson defeats Ali, Ali defeats Marciano but Marciano defeats Tyson.

Unfortunately, the who cares view is not without difficulties. We surely would like to have a voting rule which in some sense chooses the "best" outcome, where best is defined relative to the preferences of the voters. In the above example it doesn't seem to matter much whether  $A$ ,  $B$ , or  $C$  is chosen but other examples show that majority rule can lead to very bad outcomes. Far from choosing the best outcome, majority rule with pairwise voting can lead to a choice which everyone regards as worse than some other possible choice. Consider the following situation in which three voters are choosing among a list of candidates for President (this example is from Dixit and Nalebuff, 1991).

Table Two: Rankings for President by Voters L,M,R			
	<i>Voter L</i>	<i>Voter M</i>	<i>Voter R</i>
1	Happy	Grumpy	Dopey
2	Sneezy	Dopey	Happy
3	Grumpy	Happy	Sleepy
4	Dopey	Bashful	Sneezy
5	Doc	Sleepy	Grumpy
6	Bashful	Sneezy	Doc
7	Sleepy	Doc	Bashful

Now suppose we begin by voting on Happy v. Dopey  $\Rightarrow$  Dopey wins

Grumpy v. Dopey  $\Rightarrow$  Grumpy wins

Sneezy v. Grumpy  $\Rightarrow$  Sneezy wins

Sleepy v. Sneezy  $\Rightarrow$  Sleepy wins

Bashful v. Sleepy  $\Rightarrow$  Bashful wins

Doc v. Bashful  $\Rightarrow$  Doc wins.

At the end of our voting agenda Doc is the winner. But look carefully at the preferences of the three voters. Every voter would have preferred either Happy, Grumpy, or Dopey to Doc. Majority rule has led to an outcome which in the language of economists is Pareto inferior. A Pareto inferior outcome is one in



which at least one person can be made better off without making any one else worse off. Here all three voters can be made better off. Economists prefer choices to be Pareto optimal which means that no one can be made better off without making someone else worse off.

## **2.2. The cycling paradox with infinite choices**

The cycling paradox can be demonstrated very nicely using graphs. Recall that an indifference curve tells us all the combinations of two goods, say  $x$  and  $y$ , which give an individual equal utility. Typically we assume that more is better so utility is increasing in the NE direction, as in Figure 2.1.

Now suppose we have to choose among three goods: national defense, welfare, and private goods. The more we spend on national defense and welfare the more taxes have to be raised and so the less private goods are available. It's hard to draw pictures in three dimensions so we are going to suppress the private goods dimension. Preferences can then be described by circular indifference curves in two dimensions, see Figure 2.2. The optimal amount of defense and welfare programs is indicated by the bliss point - given this amount of defense and welfare the amount of private goods is also optimal. We can have more defense and welfare by moving in the NE direction but this requires higher taxes and fewer private goods which lowers

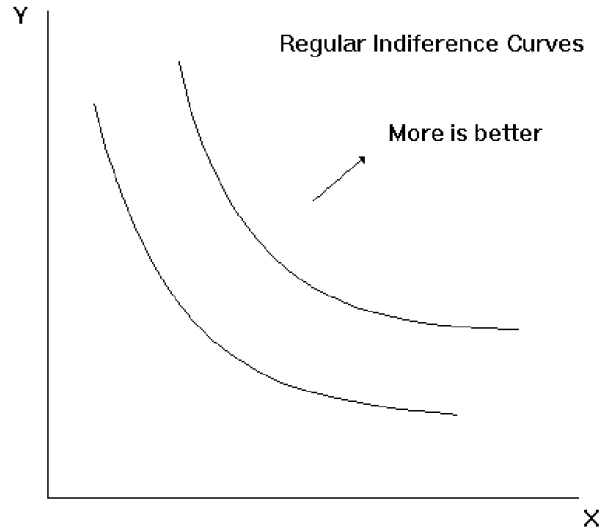


Figure 2.1: Regular Indifference Curves: Along each curve utility is constant. Utility is increasing in the NE direction.

our utility. We can have less taxes by moving in the SW direction but then we won't have an optimal amount of defense and welfare spending so our utility is less in this direction also. Moving in the NW direction gives us more defense and an ok amount of taxes but not enough social programs; similarly, moving in the SE direction gives us too little defense. Thus, the farther we move from the bliss point in any direction the lower our utility. The indifference curves tell us all the combinations of defense and welfare spending which give equal levels of utility.<sup>1</sup>

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<sup>1</sup>The indifference curves generated by this procedure will be closed loops but the loops don't have to be circular. Assuming circular indifference curves allows us to prove the theorems we are interested in using some well known geometric properties of circles.

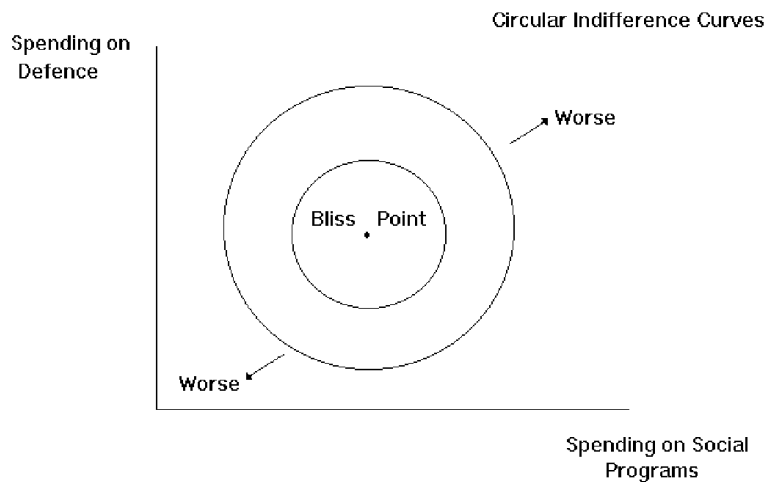


Figure 2.2: At the bliss point we have an ideal amount of defense, social programs, and implicitly private goods. Moving in any direction away from the bliss point lowers our utility.

We will use some simple geometric properties of circles to make our diagrams easier to read. Remember, that the indifference curves are concentric circles. This means that if we want to compare two points to see which has higher utility all we have to do is see which point is closer to the bliss point. We will now prove a second useful fact about circular indifference curves. Draw a line from a circle's center (what we are calling the bliss point) to the edge of the circle, a radius in other words. Now draw a tangent to the circle at this point, the tangent to the circle will always be at an angle 90 degrees to the radius. We will not prove this result but

a few examples should convince you that it is true. What we are going to prove is that if you move along the tangent in either direction away from the radius, utility is decreasing. To prove this draw another line from the bliss point and connect it to a point on the tangent line. The radius, the tangent, and the line just drawn form a triangle with the line just drawn being the triangle's hypotenuse. If we label the lines  $a, b, c$  as in Figure 2.3 then we know from Pythagoras's theorem<sup>2</sup> that the length of the hypotenuse is equal to  $\sqrt{a^2 + b^2}$ . As we move along the tangent away from the radius  $b$  is increasing (and  $a$  is constant), therefore, the length of the triangle's hypotenuse is increasing.

We have just proved that the distance from the bliss point is increasing as we move along the tangent away from the radius and we know this means that utility is decreasing. We can use these two facts to simplify our diagrams. Consider two voters with bliss points  $a$  and  $b$  as in Figure 2.4. Now join the bliss points with a line (denoted  $ab$ ) and consider any point off the line like  $z_0$ . Draw a line from  $z_0$  perpendicular to the line  $ab$  (ie. it meets  $ab$  at an angle of 90 degrees). We know from the above proof that any point which is closer to  $ab$  than  $z_0$  (along the perpendicular) is preferred by both voters to  $z_0$ . In a vote between  $z_1$  and  $z_0$ , for

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<sup>2</sup>Pythagoras's theorem says that in a right angled triangle the length of the hypotenuse squared is equal to the length of the two sides squared. If we let the length of the hypotenuse be equal to  $c$  and the lengths of the other two sides be  $a$  and  $b$  then the famous formula states that  $c^2 = a^2 + b^2$ .

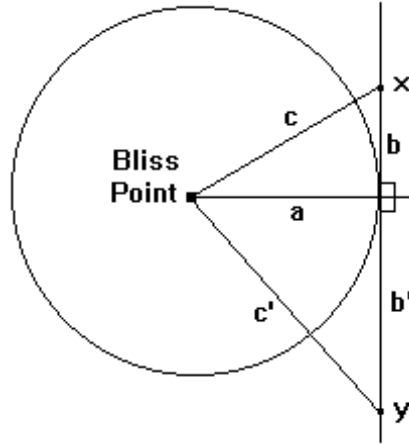


Figure 2.3: Distance from Bliss Point along a Tangent: By Pythagoras's theorem point  $x$  is closer to the bliss point than point  $y$ , point  $x$  is therefore preferred to point  $y$ .

example, voters  $a$  and  $b$  would both vote for  $z1$ .<sup>3</sup>

With our new rule we can now show the cycling paradox in two dimensions.

Consider three voters with bliss points  $a, b, c$ . Connect the bliss points with lines as in Figure 2.5.

Suppose that the status quo is point  $z0$  and point  $z1$  is brought to vote. Voters  $a$  and  $c$  prefer  $z1$  to  $z0$  and so  $z1$  will beat  $z0$  by majority rule. We indicate this by writing  $z1 \succ_{ac} z0$ . Can we find a point which beats  $z1$ ? Yes, note that the

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<sup>3</sup>For any point off the line  $ab$  we can find a point which is preferred by both voters using the procedure in the text. Any movement along the line  $ab$ , however, makes one voter worse off and the other better off. Points along the line  $ab$  are Pareto optimal points.

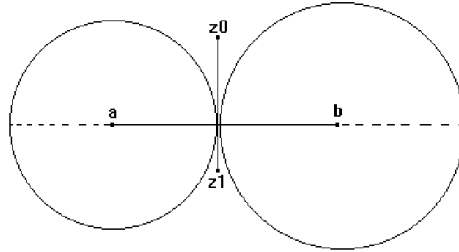


Figure 2.4:  $z_1$  is preferred by both voters to  $z_0$ .

line  $z_2z_1$  is perpendicular to line  $bc$  and along this perpendicular  $z_2$  is closer to  $bc$  than  $z_1$ . By our rule it follows that  $z_2 \succ_{bc} z_1$ . Similarly,  $z_3 \succ_{ab} z_2$  and  $z_4 \succ_{ac} z_3$ . As with our earlier example involving just three choices there is no equilibrium to this problem. If we don't limit the number of votes, majority rule is incapable of choosing a 'best' policy, voting will cycle over an infinite number of issues without ever reaching a stopping point. Suppose, however, that only four votes are taken so the final policy chosen is  $z_4$ . But everyone prefers  $z_0$  to  $z_4$ !<sup>4</sup> Majority rule can lead a group of people to choose a policy which everyone agrees is worse than another possible choice!

Figure 2.5 also indicates that cycling is not restricted to a small set of 'bizarre' preferences. Two dimensions of voting hardly seems unreasonable and yet with

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<sup>4</sup>As an exercise try showing that any point inside the triangle formed by the three Bliss points is Pareto Optimal while any point outside of the triangle is Pareto Inferior.

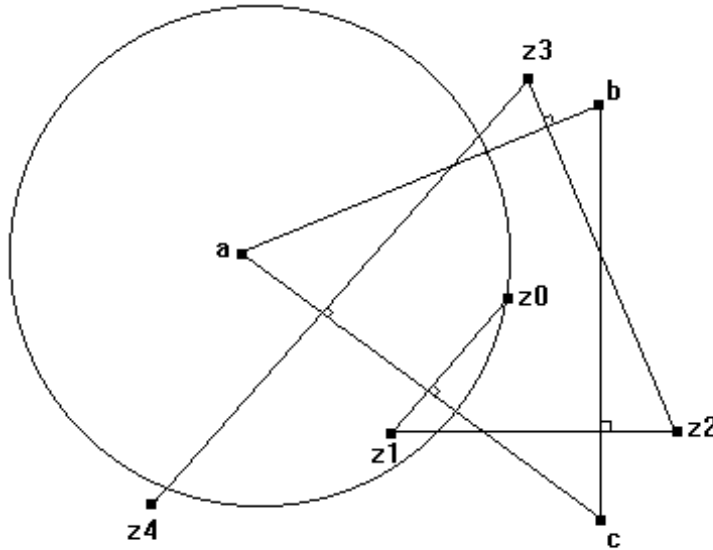


Figure 2.5: Majority Rule Can Lead to Pareto Inferior Choices:  $z1 \succ_{ac} z0$ ,  $z2 \succ_{bc} z0$ ,  $z3 \succ_{ab} z0$ ,  $z4 \succ_{ac} z0$ , but note that everyone prefers  $z0$  to  $z4$ . (One of  $a$ 's indifference curves has been drawn to indicate  $z0 \succ_a z4$ .)

two dimensions there will be a vote cycle if the lines connecting the bliss points form a triangle. The only case where cycles can be ruled out is if all the bliss points can be connected by a straight line, which seems unlikely (we will talk about this case more in chapter XX). Adding more voters or more dimensions of voting adds to the probability of cycles.

### 2.3. Implications of Cycling: Agenda Setting

We showed earlier that the following set of preferences leads to a vote cycle.

Cyclic Preferences			
	Voter One	Voter Two	Voter Three
First Best	A	C	B
2nd	B	A	C
3rd	C	B	A

Suppose that instead of letting the vote cycle go on forever we use a set of rules which prescribes a certain order of voting and a limit on the number of votes (Robert's Rules of Order, for example, is used in many Western democracies). An example of a simple rule is: pick two issues at random and vote on them, take the winner and match it against the remaining alternative, adopt the winner of the final vote.

Under the random choice rule there are three possible agendas which are illustrated in Figure 2.6. The arrows on the agendas indicate how voting proceeds given majority rule. In Agenda Two  $A$  is initially matched against  $C$ , and  $C$  wins. In the second vote,  $C$  is matched against  $B$  and is defeated. Notice that the agenda determines the winner. Preferences are the same in all three examples so the outcome is determined solely by something we would hope would be irrelevant, the order of voting.



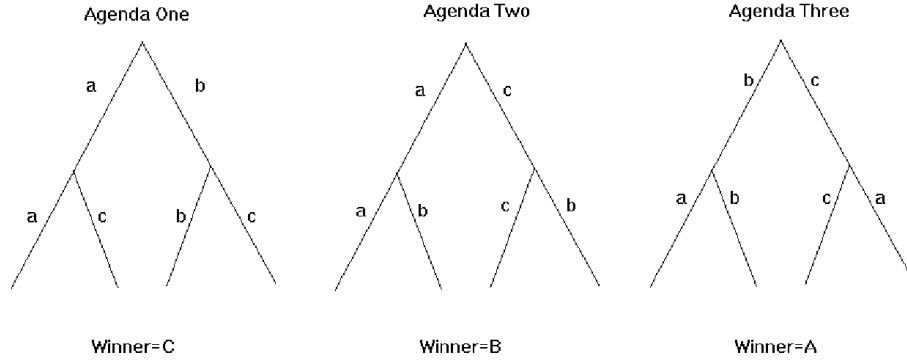


Figure 2.6: With Cyclic Preferences the Outcome is Determined by the Agenda.

. In actual practice the random choice rule is rarely used. Instead the agenda is often controlled, at least to some extent, by one or more agents. In the US House of Congress, for example, the majority party and the speaker of the House have significant control over the agenda. Of course, the minority party has some powers too and they also try to use their powers to control the agenda. Nevertheless, the majority party's power is significant. By choosing whether to use Agenda One, Two, or Three the majority party can in some circumstances advantageously manipulate a series of votes.

#### 2.4. Agenda Setting in the Extended Model

We can also show how agenda setting occurs using graphs. Suppose that  $z_0$  is the status quo and that  $b$  is the agenda setter.  $b$  can achieve his bliss point by

setting up the following agenda  $z1$  *v.*  $z0$ , winner *v.*  $z2$ , winner *v.*  $b$ . Following the agenda we have  $z1 \succ_{ac} z0$  and  $z2 \succ_{bc} z1$  and finally  $b \succ_{ab} z2$ , therefore  $b$  is the final outcome. We have drawn in part of one of  $a$ 's indifference curves in Figure 2.7 to indicate clearly that  $a$  prefers point  $b$  to  $z2$  ( $b$  is closer to  $a$ 's bliss point than  $z2$ .)

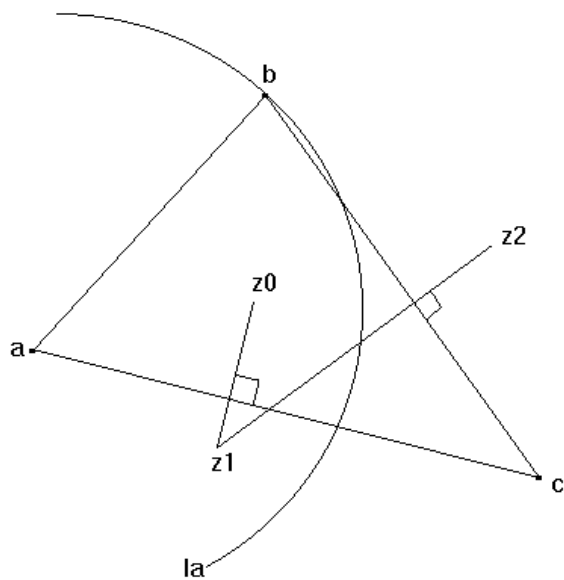
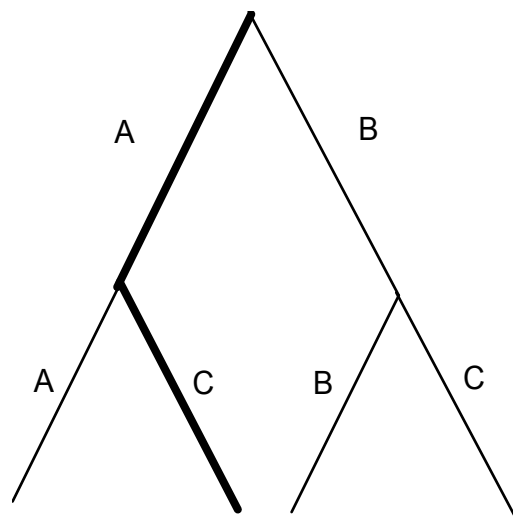


Figure 2.7: Agenda Setting in the Extended Model.  $b$  achieves his bliss point by matching  $z1$  against  $z0$ .  $z1$  wins, then  $z2$  vs.  $z1$ ,  $z2$  wins. Finally,  $b$  vs.  $z2$ ,  $b$  wins.

## 2.5. Implication of Cycling #2: The Killer Amendment

So far we have assumed that the issues up for vote are simply presented to the agents. More often, agents actively work to push the issues they are interested in onto the agenda. The opportunity to put an issue onto the agenda can be used to manipulate the final outcome. Suppose we have situation where a majority wants  $B$  and a minority wants  $C$ . Clearly in a vote between  $B$  and  $C$ ,  $B$  will win. The minority, however, might be able to add items to the agenda so that  $C$  ends up winning the final vote. If the minority can find an issue  $A$  such that  $A \succ B$  but  $C \succ A$ , then by creating the agenda in Figure ??  $C$  will win the final vote:

An issue like  $A$ , which is put onto the agenda in order to kill issue  $B$  and lose to issue  $C$ , is called a killer amendment. Killer amendments are probably hard for politicians to find. In order to work,  $A$  must beat  $B$  but lose to  $C$  - such an issue may not exist. Despite this difficulty, careful observers of the political process believe that killer amendments have been used in the past. A prominent example is the voting which occurred surrounding the adoption of the 17th Amendment to the U.S. Constitution. As part of the U.S. system of checks and balances U.S. Senators, unlike members of the House, were originally elected not by the people directly but by the state legislatures. The 17th amendment to the constitution made Senators directly elected by the people in 1913(?). The vote in favor of direct



Winner=C

Figure 2.8:

election of Senators did not win immediately, however. Many senators from the South were in favor of the direct election of Senators but they were also in favor of State's rights and they didn't want the amendment to create a precedent for the Federal control of elections. Southern senators feared that if elections came under control of the Federal government the South's policy of excluding blacks from the political process would come to an end. The fears of the southern senators were probably justified as a number of northern Republicans wanted Federal control of elections in the South in order to enfranchise blacks who would overwhelmingly vote for Republicans. To meet this difficulty, southern and northern Senators in favor of direct election hit upon a compromise. They brought a bill to the floor which would institute the direct election of senators but which also contained a provision specifically protecting the South from Federal control. This was issue *B*. But not all senators wanted to be directly elected. A prominent minority led by Senator Sutherland of Utah wanted to maintain the status quo, this was issue *C*.

Sutherland knew that if it came to a pairwise vote *B* would beat *C*. He needed to come up with a killer amendment, issue *A*. Sutherland's killer amendment was the direct election of senators but with no special provision for the South.<sup>5</sup>

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<sup>5</sup>Sutherland's amendment was modelled after a similar killer amendment brought forward

Sutherland was against the direct election of senators but his amendment is in favor. Sutherland was hoping that his own amendment would eventually fail! Sutherland's amendment did fail, exactly as he had planned. First came the vote between *A* and *B*. Sutherland and everyone else against the direct election of senators voted for *A* as did Republicans who wanted direct election and a chance to enfranchise black voters in the South. *A* beat *B*. Now *A* was matched against *C*, the status quo. Sutherland and everyone else against the direct election of senators switched their votes from *A* to *C*. Sutherland's group was joined by Southern Democrats who favored the direct election of senators but who would not vote for direct election without special protection for the South. Although *A* gained more votes than *C*, *A* was defeated and the status quo won because a constitutional amendment requires a two-thirds majority. Sutherland's killer amendment was successful.<sup>6</sup>

The story of the killer amendment shows that it is sometimes in a voter's interest to vote against the candidate or policy he most wants to win. It turns out that when there are three or more choices all voting schemes suffer from this

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by Chauncey Depew. Depew's killer amendment had killed an 1902 movement towards the direct election of Senators. See Riker (1986).

<sup>6</sup>Sutherland's success did not last long. A new Senate was able to pass the 17th amendment (without the protective clause) several months after Sutherland's brilliant politicking.

problem.<sup>7</sup> That is, in all voting schemes it will sometimes be in a voter's interests to misrepresent his preferences, the way Sutherland did by voting for the direct election of senators. In this chapter we will maintain the assumption that voters vote sincerely, in chapter XX we will look at strategic or sophisticated voting.

### 3. A Brief Review

We have shown that majority rule and pairwise voting may not yield an equilibrium outcome, ie. there may be no issue which beats all other issues in pairwise voting (no Condorcet winner). Voting may cycle among 3 (or more) issues. The Seven Dwarfs example indicates that the cycle can be “pushed out” to yield a Pareto inferior outcome. We also showed that cycling and Pareto inferior choices are possible and even likely when the issue space is composed of two continuous dimensions. Finally, we discussed some implications of cycling. If preferences are potentially cyclic, the outcome of a series of votes depends not only on what is being voted on but also on the order of voting. An agent who controls the agenda may be able to control the outcome of a series of votes. Agents sometimes put issues onto the agenda solely in order to stop other issues from winning. Voting

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<sup>7</sup>The statement in the text is a rough paraphrase of the Gibbard-Saitherwaite theorem which we will discuss more rigorously in the next chapter. There are some voting schemes where lying never pays but these either require a dictator, are trivial, or are incomplete.

for something you don't want in an early round of voting can often help you win your ultimate goal in a later round of voting.

Majority rule with pairwise comparisons is just one possible method of voting. We shall now examine a number of other voting systems. We shall find that they to are subject to all manner of surprising paradoxes.

#### **4. Another Paradox: The Failure of Positive Association**

Suppose that instead of using pairwise voting we vote on all issues at once and choose that issue which wins the most votes - this is called plurality rule. A common variant of plurality rule is plurality rule with possible runoff. In this system all the issues or candidates are voted on but if no candidate gets more than say 50% of the total vote the top two candidates enter into a runoff election. Plurality rule with runoff is used in a lot of primary elections and elections for political office including the office of Russian President. To illustrate the paradox suppose that voters have the following preferences over candidates  $A$ ,  $B$ , and  $C$ .



Table 4a: Failure of Positive Association						
# of Voters $\Rightarrow$	6	6	6	4	2	3
First	$A$	$C$	$B$	$B$	$C$	$A$
Second	$B$	$A$	$C$	$A$	$B$	$C$
Third	$C$	$B$	$A$	$C$	$A$	$B$

In the first round of the election we have the following vote tally.

$$A \Rightarrow 9$$

$$B \Rightarrow 10$$

$$C \Rightarrow 8$$

There are 27 voters in all and 50% of 27 is approximately 14, so no candidate receives more than 50% of the vote. We therefore take the two top voter getters,  $A$  and  $B$ , and have a runoff election. In the runoff election the vote tally is:

$$A \Rightarrow 15$$

$$B \Rightarrow 12$$

$A$  wins the election.

Now suppose that before the election  $A$  had given a great speech which convinced some voters that  $A$  was a better candidate than  $B$ . In particular, assume that 3 of the 4 voters who ranked the candidates  $B \succ A \succ C$  changed to  $A \succ B \succ C$  and the two voters who ranked the candidates  $C \succ B \succ A$  changed to  $C \succ A \succ B$ . The new preference rankings are:

Table 4b: Failure of Positive Association					
# of Voters $\Rightarrow$	9	8	6	1	3
First	<i>A</i>	<i>C</i>	<i>B</i>	<i>B</i>	<i>A</i>
Second	<i>B</i>	<i>A</i>	<i>C</i>	<i>A</i>	<i>C</i>
Third	<i>C</i>	<i>B</i>	<i>A</i>	<i>C</i>	<i>B</i>

In the first round of the election the vote tally is:

$$A \Rightarrow 12$$

$$B \Rightarrow 7$$

$$C \Rightarrow 8$$

Now the two top voter getters *A* and *C* go to the runoff and the vote tally is:

$$A \Rightarrow 13$$

$$C \Rightarrow 14$$

*C* wins the election! Even though *A*'s speech raised him in the voter's rankings,

*A* now loses the election. *A* was better off when fewer people thought highly of him!

This paradox is called the paradox of positive association because we would expect (and hope!) that positive changes in preferences would be associated with positive changes in outcomes. We have just shown that in one often used form of voting positive changes in preferences can lead to negative changes in outcomes.

The paradox of positive association implies that a voter who wants  $A$  to win is sometimes better off voting against  $A$ . Remember, we said above that when choosing among three or more alternatives all voting systems sometimes give voters an incentive to misrepresent their true preferences - it's not surprising therefore that we find plurality rule has this property!

A closely related paradox which can occur with plurality rule with runoff and also with majority rule and pairwise voting goes as follows. Suppose two groups of voters meet separately and each group chooses  $A$  as the best candidate. The groups now get together and vote again as a single group. One would expect that if each group 'thought'  $A$  was the best candidate then both groups choosing together would also pick  $A$ . As the following example indicates this is not necessarily the case.

Suppose the first group has 13 members with the following rankings<sup>8</sup>:

Table 5a: Preferences of Group One				
Number of Voters $\Rightarrow$	4	3	3	3
First Choice	A	B	C	C
2nd	B	A	A	B
3rd	C	C	B	A

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<sup>8</sup>The example is from Saari (1994).

In the first vote the tally is:  $A \Rightarrow 4, B \Rightarrow 3, C \Rightarrow 6$ . The two top vote getters,  $A$  and  $C$  now advance to the runoff in which the tally is:  $A \Rightarrow 7$ , and  $C \Rightarrow 6$  so  $A$  is the choice of group number one.

Suppose the second group has the following rankings:

Table 5b: Preferences of Group Two				
Number of Voters $\Rightarrow$	4	3	3	3
First Choice	A	B	C	B
2nd	B	A	A	C
3rd	C	C	B	A

In the first vote the tally is:  $A \Rightarrow 4, B \Rightarrow 6, C \Rightarrow 3$  and in the runoff:  $A \Rightarrow 7$ , and  $B \Rightarrow 6$ . Group two's choice is also  $A$ .

Both groups choose candidate  $A$ . Yet put the two groups together and even though no one changes their preferences we see that in the first vote the tally is:  $A \Rightarrow 8, B \Rightarrow 9, C \Rightarrow 9$ . Candidate  $A$  is now in *last* place! Candidates  $B$  and  $C$  enter the runoff and  $B$  ends up victorious with 17 votes to  $C$ 's 9.

## 5. Paradoxes of Irrelevant Alternatives: Dropping out and Changing Preferences

Imagine you went to the Baskin Robbins ice cream parlor and seeing that they have chocolate, vanilla, or strawberry ice cream available, you choose chocolate. Before the owner can take your order, however, you see him put a sign in the window - “Sorry, We are All Out of Vanilla”. It seems like a good principle of decision making that this should not cause you to now choose strawberry. If chocolate was the best choice when you could have had chocolate, vanilla, or strawberry then surely chocolate is the best choice when you can have only chocolate or strawberry.<sup>9</sup> Rational individuals act in this way and we might expect or hope that group decisions will be made in this way also. Under many types of voting systems, however, this is not the case.

We will demonstrate the dropping out paradox and the changing preferences paradox using the Borda Count but it is important to note that these paradoxes occur for *any* form of positional voting - we generalize our results in the next chapter. Let there be four possible candidates  $w, x, y, z$  and suppose preferences are as follows:

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<sup>9</sup>This is sometimes called value/feasibility separation (Plott, XXXX) because values shouldn't change depending on what is feasible or not.

Table 9a: The Dropping Out Paradox			
Number of Voters $\Rightarrow$	3	2	2
First Choice (3 points)	w	x	y
2nd (2)	x	y	z
3rd (1)	y	z	w
4th (0)	z	w	x

Candidate  $w$  is ranked first by three voters (3 points times 3 voters = 9 points), third by two voters (2 points), and last by two voters (0 points) for a total score of 11 points, which we write as  $w \Rightarrow 11$ . Similarly,  $x \Rightarrow 12$ ,  $y \Rightarrow 13$ , and  $z \Rightarrow 6$  points. Candidate  $y$  wins the election.

Now suppose that candidate  $z$  dies unexpectedly. Candidate  $z$  would have been ranked dead last if the election occurred so candidate  $z$ 's presence or absence should be irrelevant. If  $y$  is the best candidate among  $w, x, y, z$  then surely  $y$  should be the best candidate among  $w, x, y$ . This turns out not to be the case. The new rankings are:

Table 9b: The Dropping Out Paradox			
Number of Voters $\Rightarrow$	3	2	2
First Choice (2 points)	w	x	y
2nd (1)	x	y	w
3rd (0)	y	w	x

The point scores are  $w \Rightarrow 8$ ,  $x \Rightarrow 7$ ,  $y \Rightarrow 6$ . The winner of the election is now  $w$ ! The change is even more startling than this indicates because notice that when  $z$  was included the BC ranking among  $w, x, y$  was  $y \succ x \succ w$  but when  $z$  drops out the ranking reverses to  $w \succ x \succ y$ !<sup>10</sup>

A similar paradox occurs when voters change their preferences about ‘irrelevant’ alternatives. For example, suppose that preferences are the same as in Table 9a except that the two voters in the middle column come to believe that  $z$  is preferred to  $y$ . Preferences are thus:

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<sup>10</sup>This type of paradox also occurs with plurality rule. Suppose there are five candidates for a job in an economics department and the hiring committee uses plurality rule to rank them  $a \succ b \succ c \succ d \succ e$ . If candidate  $b$  gets a job elsewhere it is quite possible that if another vote is held the new ranking is  $e \succ d \succ c \succ a$ , even if preferences do not change! Under these circumstances is  $a$  the most preferred candidate or the least preferred? It is impossible to say.

Table 10: Failure of Independence of Irrelevant Alternatives			
Number of Voters $\Rightarrow$	3	2	2
First Choice (3 points)	w	x	y
2nd (2)	x	z	z
3rd (1)	y	y	w
4th (0)	z	w	x



Before the change the BC ranking was  $y \succ x \succ w \succ z$ . Now notice that no one has changed their preferences between the pair  $x, y$ . Everyone who thought that  $y$  was preferred to  $x$  still believes  $y$  is preferred to  $x$  and everyone who thought  $x$  was preferred to  $y$  still believes that too. Yet when we recompute the Borda Count, we find that the rankings are now  $x \succ y \succ w \succ z$ . The group changes its ‘preference’ from  $y \succ x$  to  $x \succ y$  even though no individual changes her preferences between  $x$  and  $y$ . Voting systems with either of the above properties are said to violate the independence of irrelevant alternatives condition, a condition which will become important in the next chapter.

By the way, it is worth noting that we have no way of telling whether the two voters in the middle column honestly came to believe that  $z \succ y$  or whether they simply misrepresented their preferences. If they lied about their preferences it certainly paid off. When they wrote  $x \succ y \succ z \succ w$  on their ballot the outcome was their second best choice,  $y$ . But when they wrote  $x \succ z \succ y \succ w$  the outcome was their first ranked choice,  $x$ . The fact that lying (which economists call strategic misrepresentation!) is profitable in voting games is not surprising. As noted above and discussed further in chapter xxx, all voting systems with 3 or more choices have this property.

## 6. What Does It All Mean?

What paradoxes of voting tell us is that group choice is not at all like individual choice. When an individual buys a quart of chocolate ice cream we have good reasons for thinking that he prefers chocolate to vanilla or strawberry ice cream. When a group of people buys a quart of chocolate ice cream we cannot make similar claims. Most of us know that groups don't have preferences and in this philosophical sense we know that it is illegitimate to say that the group prefers chocolate to strawberry ice cream. But the claim we are making is stronger. We might believe that groups don't have preferences in the strict philosophical sense yet also believe that group choice can be understood *as if* groups have rational (individual like) preferences. If the latter claim were true it would be a very useful fact to know. If we saw a group choosing apples rather than bananas and bananas rather than coconuts and if groups acted *as if* they had rational preferences we could predict that the group would choose apples rather than coconuts. Similarly, suppose we saw a group which has a choice of 33 flavors of ice cream chooses chocolate ice cream. If groups acted as if they had rational preferences we could conclude that the group preferred chocolate to every other flavor of ice cream and we could *predict* that if offered a choice of say chocolate, vanilla, or stawberry the

group would choose chocolate. We have given several examples of common voting schemes under which these predictions and claims are false. Not only do groups not have preferences, groups do not act as if they had rational preferences. Group choice is not at all like individual choice.

But wait - there are many types of voting schemes and we have only looked at a handful. Perhaps with more perspiration and some inspiration too we will find or create a voting system under which groups do act as if they were rational individuals. Unfortunately, there is no such voting system and there never will be, this remarkable discovery was made by the economist Kenneth Arrow in 1951. Arrow showed that there is no voting system, no matter how clever or complex, that aggregates individual preferences so that groups behave as if they were rational individuals. We turn to Arrow's Impossibility Theorem in the next chapter.

One last point before turning to Arrow's theorem. If something is impossible it can't be bad. Groups don't behave like individuals and this is initially surprising but not necessarily lamentable. Some of the paradoxes we have discussed *do* show that certain types of voting system are devastatingly flawed. That majority rule with pairwise comparisons could lead to Pareto inferior choices is a serious objection to majority rule. It is less obvious that we should want a group to

choose  $y$  from  $w, x, y$  when the group chose  $y$  from  $w, x, y, z$ . We initially expect this to be the case but once we know groups don't necessarily behave in this way is this cause for great concern? Voting theory is filled with more than paradoxes. Some voting systems are better than others and in chapter xxx we will try to make some positive progress in this direction.