A Geometric Proof of the Neutrality Theorem

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The second fundamental theorem of welfare economics states that under appropriate conditions every Pareto optimal outcome can be achieved as a competitive equilibrium by suitable lump-sum transfers of wealth. The second theorem implies that a central planner can implement whatever outcome he regards as best, simply by transferring income among individuals. Intuition suggests that the second theorem, or something like it, should also hold for public goods. Of course, voluntary provision of a public good will typically be inefficient because of the free rider problem. But if the central planner wanted to increase the amount of a public good in a competitive (voluntary) equilibrium could he not transfer income to those individuals who valued the public good most? Economists have been surprised to discover that this procedure will not work. For public goods a nearly opposite result to the second theorem holds. The neutrality theorem, first discovered by Warr (1983), states that under appropriate conditions lump-sum transfers of income from one person to another will cause *no* change in the amount of the public good provided. A closely related result, called the crowding out theorem, shows that the central planner cannot increase the amount of the public good via lump-sum taxes and public provision. For every unit of the public good provided by the central planner, voluntary provision will fall by one unit. Since voluntary provision of public goods is widespread, this result has aroused considerable interest.

In this paper, I prove the neutrality and crowding out theorems using a simple triangle diagram due to Dolbear (1967), henceforth called the Dolbear triangle. The widely known Edgeworth box diagram can be used to explain almost every significant theorem involving general equilibrium with private goods. The Dolbear triangle is equally useful in the analysis of general equilibrium with public goods. A secondary purpose of the paper, therefore, is to present the diagram and some of its applications.

A more precise statement of the neutrality theorem is now given. Consider a model with one private and one public good and n consumers. Consumer i has wealth, w_i, which he allocates between an amount, y_i, of the private good and a contribution, g_i, to the public good. The total amount of the public good is $G = f(\sum_{i=1}^{n} g_i)$ The consumer has utility function $U_i(y_i, G)$. Under these conditions the following theorem can be proved (Theorem One of Bergstrom, Blume and Varian (1986)).

Neutrality Theorem.

Assume that consumers have convex preferences and that contributors are originally in a Nash equilibrium. Consider a redistribution of income among contributing consumers such that no consumer loses more income than his original contribution. After the redistribution there is a new Nash equilibrium in which every consumer changes the amount of his gift by precisely the change in his income. In this new equilibrium, each consumer consumes the same amount of the public good and the private good that he did before the redistribution.

An important assumption of the neutrality theorem is that the set of contributing agents is not changed by the redistribution. The neutrality theorem will not hold when transfers cause contributors to become non-contributors or vice-versa. Similarly, the crowding out theorem holds only when the central planner restricts the tax base to contributing agents and does not tax any of these agents more than their contribution. The Nash assumption means that we will look for an equilibrium in which each agent maximizes utility by choosing g_i, assuming that all other agents hold their contributions constant. The Nash assumption can be relaxed to allow for a variety of conjectural variation models but the neutrality theorem will not, in general, carry over to more complicated bargaining models.[1]

We will prove the neutrality and crowding out theorems in a two person economy. In Figure One, the length of the Y axis gives the total amount of wealth in society, measured in terms of the private good. Just as in an Edgeworth box, Adam's endowment is read in an upwards direction and David's endowment is read in a downwards direction. An endowment point such as E indicates that Adam is endowed with A_oE units of private good and David is endowed with D_oE units of the private good. The amount of the public good is measured along the X axis. (Since both Adam and David consume the same amount of the public good we measure both their consumption levels in the easterly direction - this explains the difference between the Dolbear triangle and the Edgeworth box.) The hypotenuse of the triangle is the production possibilities curve. It is assumed that the production function is linear and available to both parties, linearity is convenient but not necessary. Let the slope of the production possibilities curve be $-b$, which indicates that b units of the private good must be given up in return for one unit of the public good. A point in the triangle, such as Q, tells us everything about the allocation of resources in this society. Adam's private wealth is given by the distance G_1Q, David's private wealth is given by the distance ZQ. The total amount of the public good is A_oG_1. By comparing private wealth at point Q with endowment wealth we can see that Adam contributed EY_A to the public good and David contributed EY_D (note that $D_oY_D=ZQ$). Adam's and David's contributions together are enough to produce G_1 units of the public good.

In Figure Two, we add indifference curves and budget constraints. Adam's indifference curves and budget constraint have the usual shape and location. Assume, for example, that David contributes nothing towards the public good. Starting at the endowment point, Adam can produce the public good using the common production function. His budget constraint, therefore, is the line EE'' running parallel to the production possibilities curve. At the utility maximizing point A^* Adam has private good consumption in the amount $A_oY_A^*$ and public good consumption of G^*_A. David also consumes G^*_A of the public good but retains all of his private goods ($ZA^*=D_oE$).

Assume that Adam contributes nothing to the public good, then David's budget constraint is given by the line EE'. At the utility maximizing point D^*, David's private wealth is given by TD^*. Moving along EE' David's private wealth is falling and the amount of the public good is increasing at exactly the rate given by the production function (i.e. $-b$ per unit of the public good). Line EE' is David's budget constraint because it gives all the combinations of private and public good that are available to David given his income and the production possibilities curve.

[1] In a conjectural variation model each agent believes that his contribution will induce others to change their contributions by some amount a, where a may be a function of other parameters, see Bergstrom, Blume and Varian

Thus, a budget constraint (for David) with slope 0 in the triangle diagram corresponds to a budget constraint with slope *-b* in the regular indifference curve diagram. Generalizing, a slope of *m* in the triangle diagram corresponds to a slope of *-m-b* in the regular indifference curve diagram.[2]

David wants more income and more of the public good, so indifference curves towards the bottom right of the diagram represent higher utility levels. David's indifference curves are easily drawn so long as one constraint is kept in mind. If we move along the indifference curve towards the right the amount of the public good is increasing; to maintain indifference David's income must be falling - thus, the distance between the indifference curve and the triangle's hypotenuse must decrease as we move along the curve towards the right.

For any amount of public good production chosen by David, Adam will face a budget line parallel to the triangle's hypotenuse and beginning at the point on *EE'* chosen by David. Suppose David were to choose point *E'*. Adam would then face the budget line *E'F* and would choose optimum point A^*_1.[3] Similarly, if Adam were to spend all of his income on the public good, David would face the budget constraint *E"F* and optimize at point D^*_1.

For every choice of public good production by David, Adam has a corresponding optimum. Adam's optimization points trace out his reaction function and similarly for David. These are drawn in Figure Three.[4] Adam's and David's reactions are consistent with each other only where the reaction functions cross, which is at the Nash equilibrium.

The neutrality theorem can now be proved with ease. Suppose in Figure 3 that *E* is the initial endowment point. Now imagine the central planner takes income from Adam and gives it to David so the new endowment point is at E_1. It is evident that this does not shift the reaction functions and hence the Nash equilibrium remains a Nash equilibrium and the amount of the public good provided does not increase. At *NE*, Adam's private wealth is A_0Y_A. Thus, if the endowment point is at *E*, Adam contributes EY_A to the public good but if the central planner

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(1986).

[2] To check the correspondence formula for slope, try a slop of *-b* in the triangle diagram - what does this imply about the cost of the public good to David?

Dolbear (1967), and Shibata (1971) give folding, twisting, or turning instructions which show how the regular indifference curve diagram can be mapped into the triangle diagram.

[3] Since Adam cannot tax David, he cannot choose a point along D_oE'. Adam's optimum may be a corner solution at *E'*.

[4] To derive David's reaction function above E and Adam's reaction function below E and interior to the budget constraint, we vary the endowment point.

confiscates EE_1, Adam contributes only E_1Y_A. In other words, Adam reduces his contribution by EE_1, exactly the amount which is confiscated by the central planner. Similarly, when the endowment point is E, David's private income is D_0Y_D (=ZNE) and his contribution Y_DE. When the Central planner transfers EE_1 units of income to him he increases his contribution by exactly EE_1 so his total contribution is Y_DE_1.

Using the Dolbear triangle we can also see what happens if the central planner taxes a contributor more than his initial contribution. Recall that if the endowment point is E, Adam contributes EY_A to the public good. If the central planner taxes Adam more than EY_A, the new equilibrium is along David's reaction function below the point NE.[5] Notice that the supply of the public good must increase if the public good is a normal good. Thus, the central planner can use lump sum transfers to increase the amount of the public good produced in a voluntary equilibrium but only if the transfers are larger than the initial voluntary contributions. It follows that a transfer from a non-contributor to a contributor will increase the amount of the public good provided.

The crowding out theorem is illustrated in Figure 4. Suppose the central planner taxes Adam some amount T_A (which is less than Adam's initial contribution) and David some amount T_D (also less than his initial contribution) such that the total is D_oT. The central planner then spends all of D_oT to purchase A_oG^{CP} of the public good. The effect of this transfer is to shift the private goods axis to the right. The total private wealth of our society is now $A'_oD'_o$, and G^{CP} of the public good is provided by the central planner. Adam's and David's origins shift to the right and their reaction functions become truncated at G^{CP} because even should they so desire they cannot consume less than G^{CP} of the public good. Despite the central planner's efforts, the total amount of the public good produced does not change. Adam and David reduce their contributions by a dollar, for every dollar the central planner spends.

The crowding out theorem follows directly from the neutrality theorem. Suppose that the central planner wants to raise taxes of amount $T_A + T_D$ and use the proceeds to produce the public good. Instead of proceeding as before, however, he first taxes Adam T_A. And, instead of producing the public good himself, he transfers T_A to David. From the neutrality theorem we know that David will use all of T_A to produce the public good. But this was exactly what the

[5] Similarly, if the central planner taxes David more than his initial contribution the new equilibrium is found on Adam's reaction function to the right of the NE.

planner would have done with the funds had he kept them. The planner, therefore, is equally happy using T_A to produce the public good himself or transferring T_A to David. By the same reasoning if the planner taxes David T_D and transfers the funds to Adam, Adam will use all of T_D to produce the public good - which is exactly what the planner was going to do with the funds. The central planner's "produce your own" plan is thus equivalent to a series of bilateral transfers. But we know from the neutrality theorem that these transfers will not increase the total amount of the public good produced and therefore, however conducted, the central planner's operation will fail.

Although the central planner cannot increase the amount of the public good via lump sum taxes he can do so using combinations of subsidies and transfers. Such subsidy/transfer schemes will in general be quite complicated and may be beyond the informational abilities of the central planner. An example of what is required is given in the appendix.

Conclusions

The Dolbear triangle is a simple, yet very powerful, tool for analyzing public goods provision in general equilibrium. Shibata (1971) uses the triangle to prove the Samuelson conditions for optimal public good provision (see also this appendix) and to demonstrate the often overlooked point that in general there is a different Pareto optimal level of public good provision for every different income distribution. The Dolbear triangle can also be used to understand the theory of positive and negative externalities.[6] E. O. Olsen uses the triangle in precisely this way to shed light on several subtle theorems in the theory of negative externalities (Olsen 1979), and in the theory of positive externalities with particular attention given to the theory of optimal income redistribution (Olsen 1981).

References

[1] Bergstrom, T. C., L. Blume, and H. Varian. 1986. On the Private Provision of Public Goods. *Journal of Public Economics* 29:25-49.

[6] Assume that some action of Adam's (measured along the *X* axis) generates a positive or negative externality for David. The triangle diagram then gives us a simple general equilibrium model of externalities.

[2] Dolbear, T. F. 1967. On the Theory of Optimum Externality. *American Economic Review* 57:90-103.

[3] Olsen, E. O. 1979. the Simple Analytics of External Effects. *Southern Economic Journal* 45 (3):847-54.

[4] Olsen, E. O. 1981. The simple analytics of the externality argument for redistribution. In *Economic Perspectives: An Annual Survey of Economics*, ed. M. B. Ballabon, 155-73. Vol. 2. New York: Harwood Academic Publishers.

[5] Shibata, H. 1971. A Bargaining Model of the Pure Theory of Public Expenditure. *Journal of Political Economy* 79 (1):1-29.

[6] Warr, P. 1983. The private provision of a public good is independent of the distribution of income. *Economic Letters* 13:207-211.

A. Appendix

In Figure 5 we show how the central planner may implement an efficient allocation of public goods. This diagram also allows us to give a simple proof of the Samuelson (1954) conditions for the efficient provision of public goods. Note that any point of tangency between Adam's and David's indifference curves is an efficient allocation – any movement from a tangency point makes at least one person worse off. The contract curve, *C'C*, describes all efficient allocations. If Adam were to own all of the income in society he would choose point *C'* (Adam's dictatorial optimum) and if David were to own all of the income in society he would choose point *C* (David's dictatorial optimum).[7] Since a movement from either of these points makes at least one person worse off these points must be on a contract curve.

Suppose the central planner wishes to implement the efficient allocation *P*, which both Adam and David prefer to the Nash equilibrium (*NE*). The line *TT'* is the tangency line through point *P*. Adam's budget constraint, *EE"*, which runs parallel to the triangle's hypotenuse, gives the trade-off between private income and the public good and has slope $-b$. Now consider the

[7] The points *C'*, *C* are not necessarily at the tips of the respective reaction functions. Given an endowment of *E*, Adam's reaction function tells us that should David spend all of his income on the public good Adam will choose point *C'*. Since Adam chooses to *add* to David's contribution ($C' > E'$) it is clear that if Adam were given all of David's endowment he would still choose *C'*. Thus, *C'* is Adam's "dictatorial" optimum. If Adam's dictatorial optimum were less than *E'* the contract curve would end at this optimum but the reaction function would end at *E'*. Similar, considerations hold for the location of David's reaction function and dictatorial optimum.

truncated line through point E which runs parallel to line TT'. Movement along this line indicates that one unit of the public good can be had for less than b units of private income. The difference between the slope of EE'' and TT' thus represents a subsidy to Adam in the amount of s_A. (The slope of line TT' can thus be written $-(b-s_A)$.) Similarly, the difference between the slope of the line EE' and the slope of line TT' represents a subsidy to David. Since the slope of line EE' is zero we have that $s_D = 0-$ slope TT', which implies that the slope of line TT' can also be written as $-s_D$. Putting the above together, the efficient allocation P can be implemented via subsidies to Adam and David of s_A and s_D respectively plus a lump sum transfer of income to Adam of $T - E$.

The Samuelson (1954) conditions can be easily demonstrated. In equilibrium, Adam's marginal rate of substitution is equal to the slope of line TT', i.e., $mrs_A^{YG} = -(b - s_A)$. Now recall that if David's budget constraint has slope m in the triangle diagram then in the regular diagram it has slope $-m-b$. Thus, at the tangency point in the triangle diagram, $mrs_D^{YG} = -(-s_D - b)$. Rearranging we have $-s_D = -mrs_B^{YG} - b$. But $-(b - s_A) = s_D$ or substituting and rearranging $mrs_A^{YG} + mrs_D^{YG} = -b$ which is Samuelson's condition for the optimal provision of a public good.

Figure 1

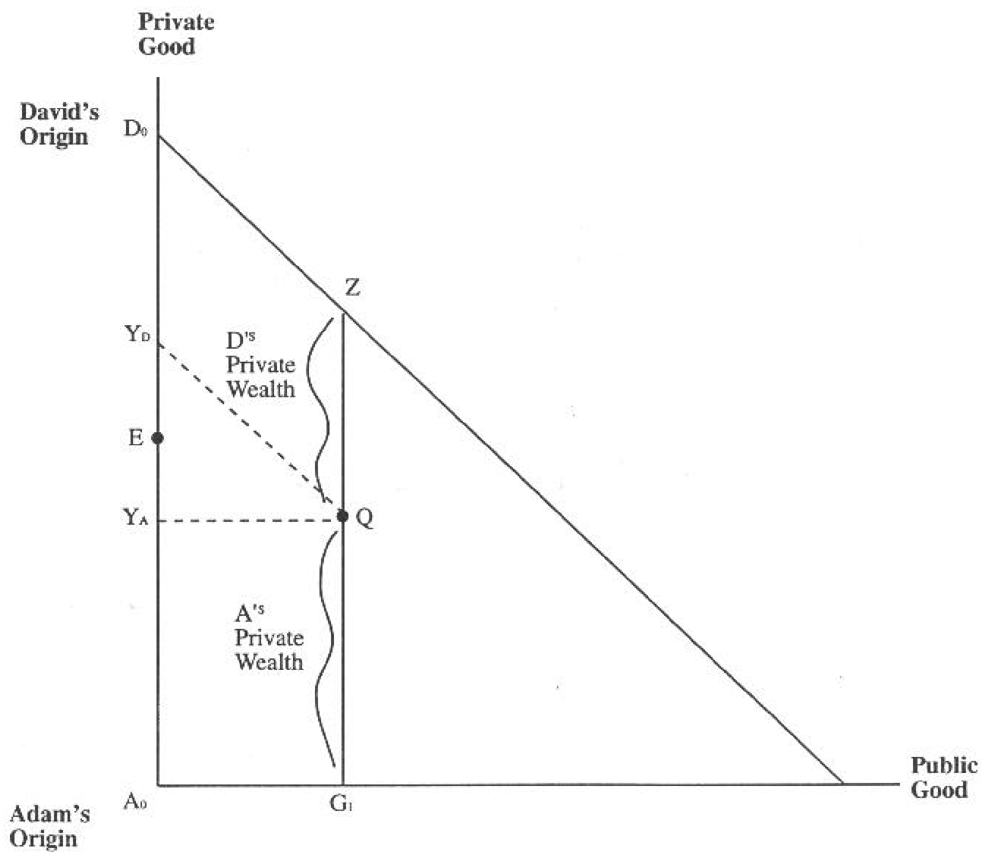
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Figure 2

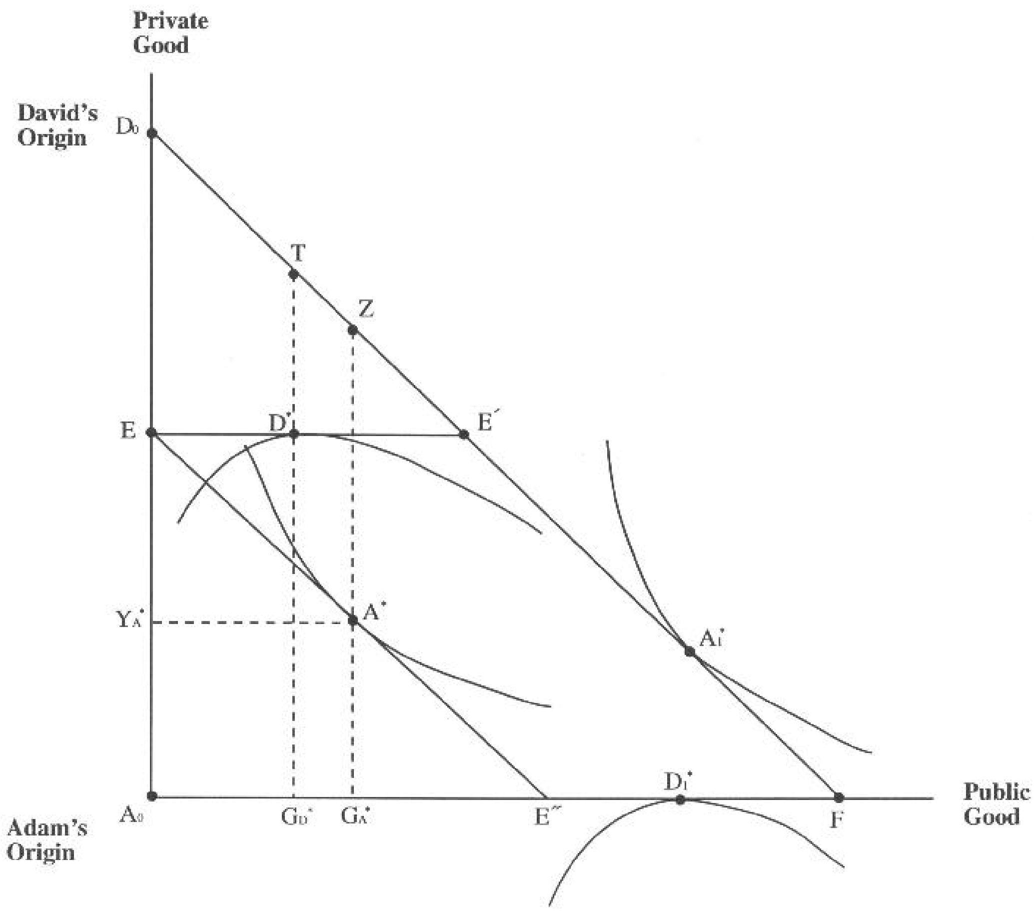

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Figure 3

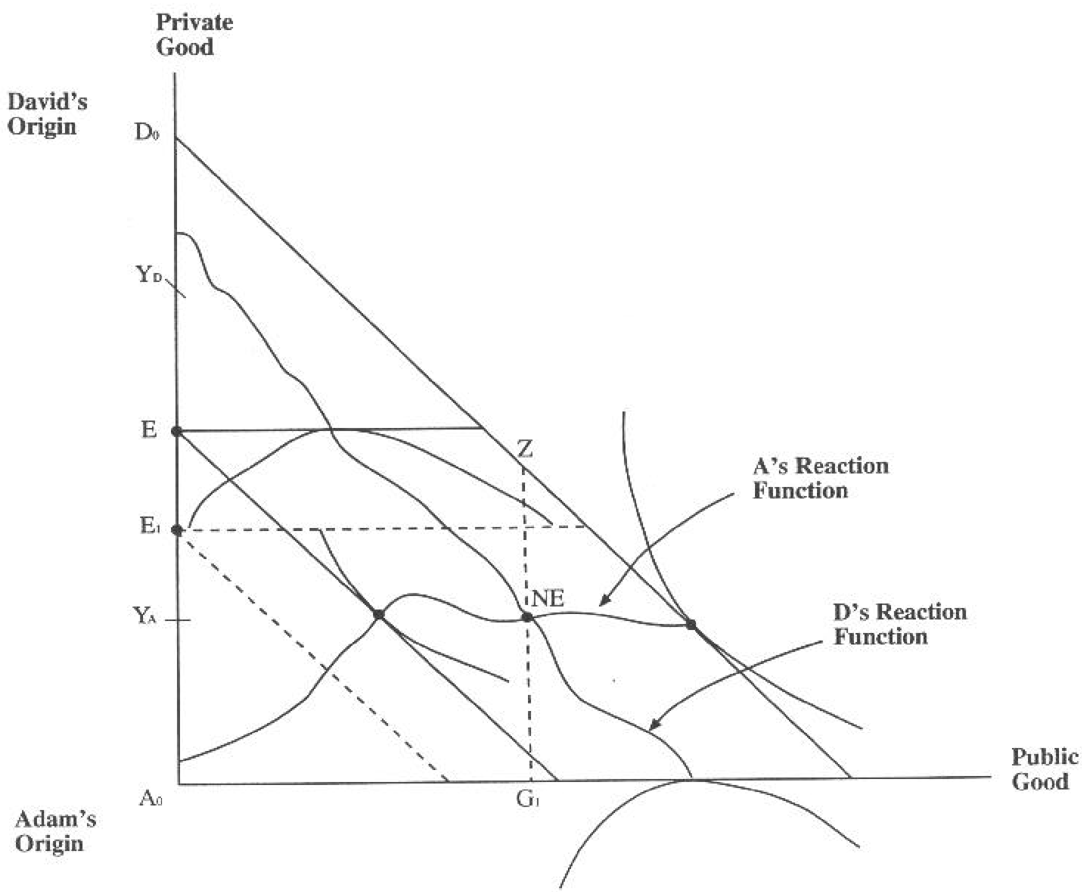

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Figure 4

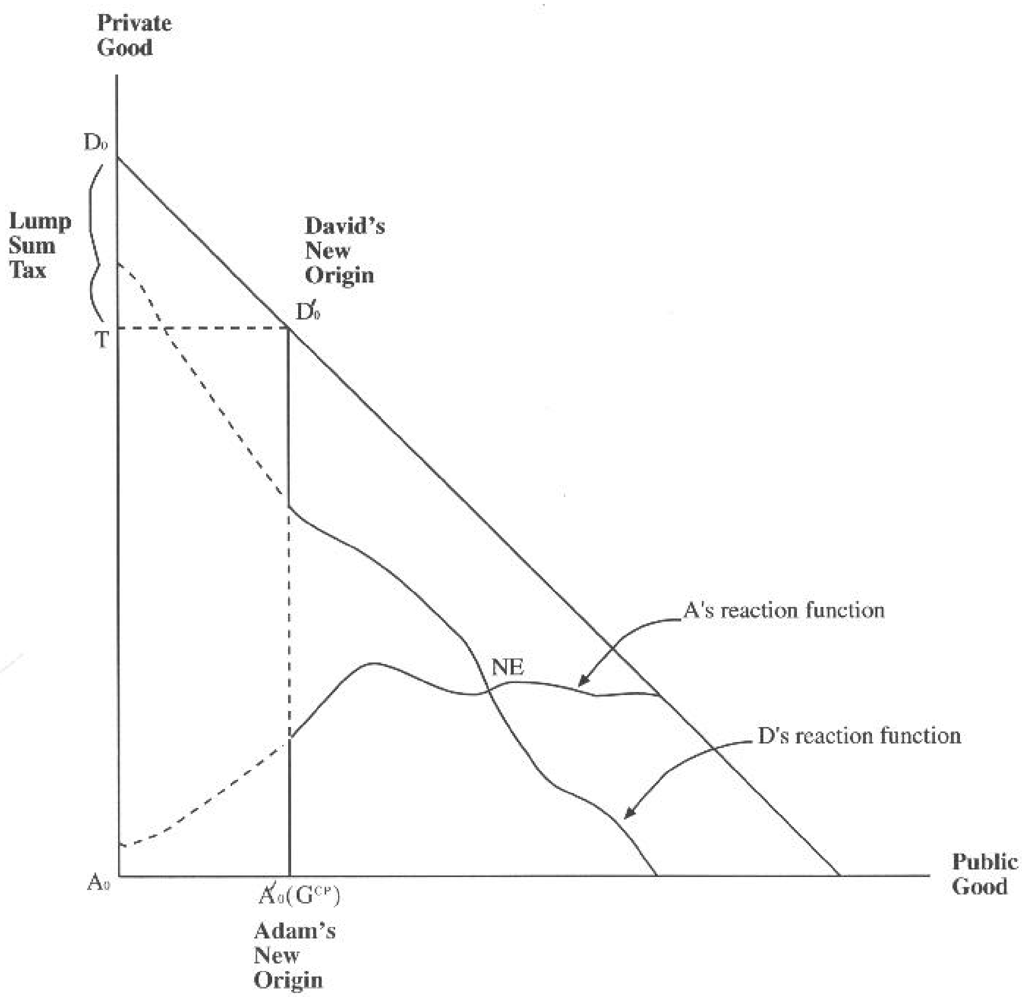
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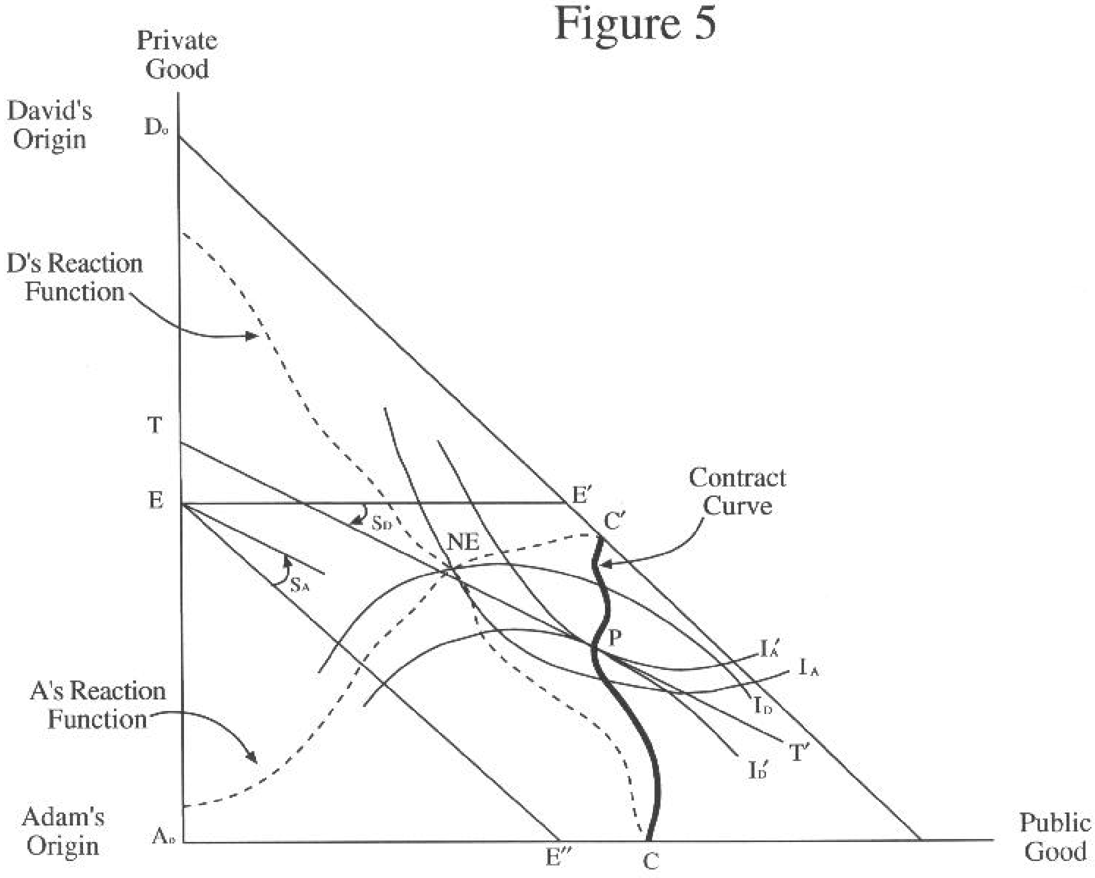
Figure 5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