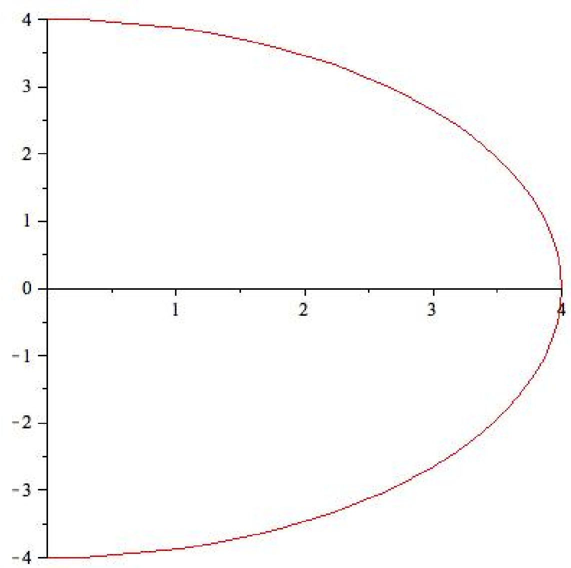
This is a two hour exam. Show all work for full credit. No electronic devices or formula sheets are allowed.

Problem 1 (10 points):

a) (5 points) Find the value of $\dfrac{dy}{dx}$ in terms of t for $x = 4\sin t, y = 3\cos t$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{3\sin t}{4\cos t} = -\frac{3}{4}\tan t$$

b) (5 points) Graph $x = 4\sin t, y = 4\cos t$ for $0 \le t \le \pi$ and indicate the direction of motion.



The motion is clockwise.

Problem 2 (10 points):

Use Taylor series to evaluate $\lim_{x\to 0} \dfrac{e^{bx} - 1}{x}$ for a real parameter b.

$$e^{bx} = \sum_{k=0}^{\infty} \frac{(bx)^k}{k!} \implies \lim_{x\to 0} \frac{e^{bx}-1}{x} = \lim_{x\to 0}\left[\frac{1}{x}\left(\sum_{k=0}^{\infty}\frac{b^k x}{k!} - 1\right)\right] = \lim_{x\to 0}\sum_{k=1}^{\infty}\frac{b^k x^{k-1}}{k!}$$

$$= \lim_{x\to 0}\left(b + \frac{b^2 x}{2} + \frac{b^3 x^2}{3!} + ...\right) = b$$

Problem 3 (10 points):
Find the linear and quadratic approximations to e^x centered at $a = 1$.

$$p_1(x) = e^1 + e^1(x-1) = ex$$

$$p_2(x) = e^1 + e^1(x-1) + \frac{e^1}{2}(x-1)^2 = ex + \frac{e}{2}(x-1)^2$$

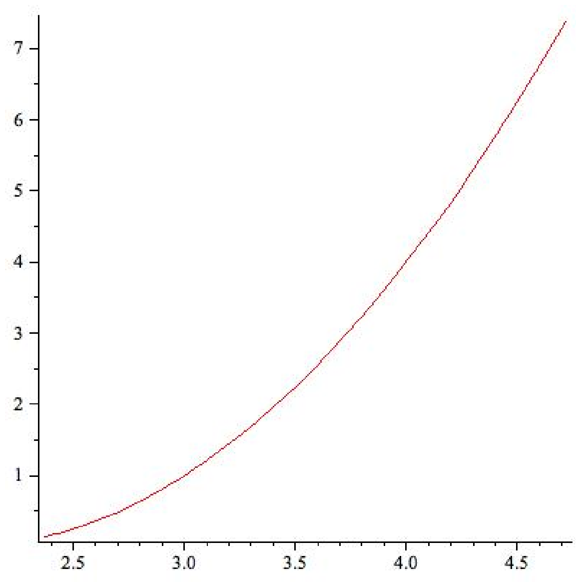
Problem 4 (10 points):
Use Taylor series to find the first four nonzero terms of a series which is equal to e^3.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \implies e^3 = \sum_{k=0}^{\infty} \frac{3^k}{k!} = 1 + 3 + \frac{9}{2} + \frac{27}{6} + ...$$

Problem 5 (10 points): Consider the parametric equations $y = e^{2t}, x = e^t + 2$
a) (5 points) Eliminate the parameter t to obtain an equation in x and y.

$$x = \sqrt{y} + 2 \implies y = (x-2)^2$$

b) (5 points) Describe the curve and indicate the positive orientation.



It is half a parabola and moves up and to the right.

Problem 6 (10 points):

a) (5 points) Find the function represented by the power series $\sum_{k=0}^{\infty}\left(\frac{k!+3}{k!}\right)x^k$

$$\sum_{k=0}^{\infty}\left(\frac{k!+3}{k!}\right)x^k = \sum_{k=0}^{\infty} x^k + 3\sum_{k=0}^{\infty}\frac{x^k}{k!} = \frac{1}{1-x} + 3e^x$$

b) (5 points) Find the interval of convergence for the power series $\sum_{k=0}^{\infty}\frac{(-1)^k x^{3k}}{27^k}$

Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{\left|\frac{x^{3k}}{27^k}\right|} = \left|\frac{x^3}{27}\right| < 1 \implies -3 < x < 3.$

At x=3: $\sum_{k=0}^{\infty}(-1)^k$ diverges by the Divergence Test.

At x=-3: $\sum_{k=0}^{\infty} 1^k$ also diverges by the Divergence Test.

Therefore the interval of convergence is (-3,3).

Problem 7 (10 points):
Estimate the remainder for the nth order Taylor polynomial for the function $\cos x$ at a=0.

Since $-1 \le \cos x^{n+1} \le 1$, we can take $M = 1$ and $|R_n| \le \frac{M\,|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$

Problem 8 (10 points):
Find the Taylor series expansion for the function e^x centered at $a = \ln 3$

$f^{(k)} = e^{\ln 3} = 3$ for all $k = 1, 2, ...$ so the Taylor series is $\sum_{k=0}^{\infty}\frac{3}{k!}(x-\ln 3)^k$

Problem 9 (10 points):
Find a power series representation centered at 0 for $f(x) = \dfrac{1}{(1-x)^2}$. Give the interval of convergence for the resulting series.

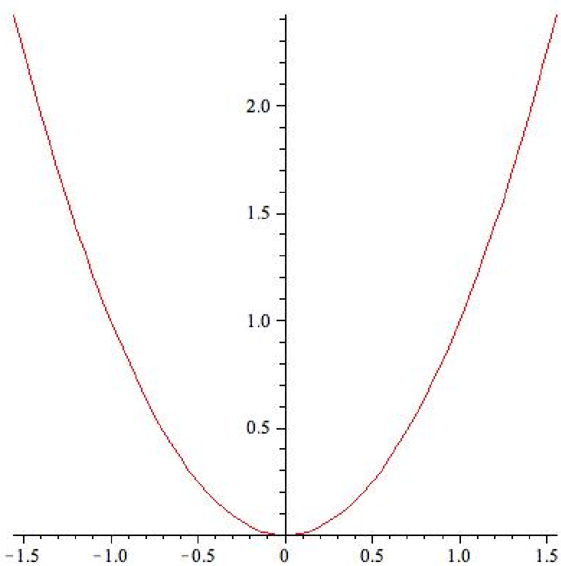
If $g(x) = \dfrac{1}{1-x}$, then $g'(x) = \dfrac{1}{(1-x)^2} = f(x)$.

Now $g(x) = \sum_{k=0}^{\infty} x^k for\, |x| < 1 \implies f(x) = \sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty}(k+1)x^k$

for $|x| < 1$ and the interval of convergence is (-1,1).

Problem 10 (10 points):
a) (5 points) Convert the polar equation $r = \sin\theta \sec^2\theta$ to Cartesian coordinates and graph the curve.

$r\cos\theta = \dfrac{\sin\theta}{\cos\theta} \implies x = \tan\theta = \dfrac{y}{x} \implies y = x^2$



b) (5 points) Write the point $(x, y) = (-1, 0)$ in two different ways using polar coordinates.

$(1, \pi)$

$(1, -\pi)$

Extra Credit (5 points):

Let $i = \sqrt{-1}$. Expand the complex exponential functions $e^{i\theta}$ and $e^{-i\theta}$ in Maclaurin series and show that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

First, observe that $i^2 = -1, i^3 = -i, i^4 = 1$, etc.

Using Maclaurin series, we have

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + ...$$

$$e^{-i\theta} = \sum_{k=0}^{\infty} \frac{(-i\theta)^k}{k!} = 1 - i\theta + \frac{(i\theta)^2}{2} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + ...$$

Now

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1 + 2\frac{(i\theta)^2}{2} + 2\frac{(i\theta)^4}{4!} + ...}{2} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} = \cos\theta$$

and

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{2i\theta + 2\frac{(i\theta)^3}{3!} + 2\frac{(i\theta)^5}{5!} + ...}{2i} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} = \sin\theta.$$