This is a two hour exam. Show all work for full credit. State which theorems or tests you are using. No electronic devices or formula sheets are allowed.

Problem 1 (10 points): Evaluate the series $\sum_{k=1}^{\infty} \frac{2^k}{e^k}$ or state that it diverges.

This is a geometric series with $a = \frac{2}{e}$ and $r = \frac{2}{e} < 1$. Thus it converges to $\frac{a}{1-r} = \frac{\frac{2}{e}}{1-\frac{2}{e}} = \frac{2}{e-2}$

Problem 2 (10 points): Determine whether $\sum_{k=1}^{\infty} \frac{\pi^k}{k!}$ converges or diverges.

We use the Ratio Test: $r = \lim_{k\to\infty} \frac{\frac{\pi^{k+1}}{(k+1)!}}{\frac{\pi^k}{k!}} = \lim_{k\to\infty} \frac{\pi}{k+1} = 0 < 1$.

Therefore, $\sum_{k=1}^{\infty} \frac{\pi^k}{k!}$ converges.

Problem 3 (10 points): Evaluate the series $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$ or state that it diverges.

This is a telescoping series with nth partial sum $S_n = 1 - \frac{1}{n+1}$.

Since $\lim_{n\to\infty} S_n = 1$, $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$ converges to 1.

Problem 4 (10 points): Find the limit of the following sequences or determine that the limits do not exist.

(a) (5 points) $a_n = \sin\left(\frac{n\pi}{2}\right)$

The terms of the sequence oscillate forever between the values -1,0,1. Therefore $\lim_{n\to\infty} a_n$ does not exist.

(b) (5 points) $a_n = \left(1 + \frac{3}{n}\right)^n$

This limit $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n$ is of the indeterminate form 1^∞.

We take the log, use L'Hôpital's Rule, and exponentiate the result.

Thus $\lim_{n\to\infty}\ln\left(1+\frac{3}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{3}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{-\frac{3}{n^2}}{-\frac{1}{n^2}} = 3.$

Therefore $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$

Problem 5 (10 points):

(a) (5 points) Determine whether $\sum_{k=1}^{\infty} ke^{-k^2}$ converges or diverges.

The function $f(x) = xe^{-x^2}$ is positive for $x \geq 1$ and $f'(x) = e^{-x^2} - 2x^2e^{-x^2} < 0$ for $x \geq 1$ so $f(x)$ is also decreasing and continuous for $x \geq 1$.

Now the Integral Test tells us $\sum_{k=1}^{\infty} ke^{-k^2}$ and $\int_1^\infty xe^{-x^2}$ both converge or both diverge.

Let $u = -x^2$ so that $du = -2xdx, u(1) = -1, u(\infty) = -\infty$. Then $\int_1^\infty xe^{-x^2} = -\frac{1}{2}\lim_{b\to\infty}\int_{-1}^{-b} e^u du = -\frac{1}{2}\lim_{b\to\infty}(e^u|_{-1}^{-b} = \frac{1}{2e}$ so that the $\int_1^\infty xe^{-x^2}$ converges.

Therefore, $\sum_{k=1}^{\infty} ke^{-k^2}$ also converges.

(b) (5 points) How many terms of the series $\sum_{k=1}^{\infty}\frac{3}{k^4}$ must be summed in order to obtain an approximation that is within 10^{-3} of the exact value of the series?

The remainder $R_n \leq \int_n^\infty \frac{3}{x^4}dx = 3\lim_{b\to\infty}\left(-\frac{1}{3x^3}\right|_n^b = \frac{1}{n^3}$. Setting $R_n = 10^{-3}$ yields $\frac{1}{n^3} = \frac{1}{10^3} \implies n = 10$ terms of the series.

Problem 6 (10 points): Determine whether $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{2}{k}\right)^{k^2}$ converges or diverges.

We use the Root Test: $\displaystyle\rho = \lim_{k\to\infty}\sqrt[k]{a_k} = \lim_{k\to\infty}\sqrt[k]{\left(1+\frac{2}{k}\right)^{k^2}} = \lim_{k\to\infty}\left(1+\frac{2}{k}\right)^{k}$.

This limit $\displaystyle\lim_{k\to\infty}\left(1+\frac{2}{k}\right)^{k}$ is of the indeterminate form 1^∞.

We take the log, use L'Hôpital's Rule, and exponentiate the result.

Thus $\displaystyle\lim_{k\to\infty}\ln\left(1+\frac{2}{k}\right)^{k} = \lim_{k\to\infty}k\ln\left(1+\frac{2}{k}\right) = \lim_{k\to\infty}\frac{\ln\left(1+\frac{2}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty}\frac{-\frac{2}{k^2}}{-\frac{1}{k^2}} = 2.$

Therefore $\displaystyle\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n} = e^2 > 1$ and $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{2}{k}\right)^{k^2}$ diverges.

Problem 7 (10 points): Determine whether $\displaystyle\sum_{k=1}^{\infty}\frac{k^2+3k+5}{k^3+2k+1}$ converges or diverges.

We compare $\displaystyle\sum_{k=1}^{\infty}\frac{k^2+3k+5}{k^3+2k+1}$ with the harmonic series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ using the Limit Comparison Test:

$$\lim_{k\to\infty}\frac{\frac{k^2+3k+5}{k^3+2k+1}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{k^3+3k^2+5k}{k^3+2k+1} = 1 \text{ and } 0 < 1 < \infty$$

so $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ and $\displaystyle\sum_{k=1}^{\infty}\frac{k^2+3k+5}{k^3+2k+1}$ either both converge or both diverge.
We know the harmonic series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, so $\displaystyle\sum_{k=1}^{\infty}\frac{k^2+3k+5}{k^3+2k+1}$ must also diverge.

Problem 8 (10 points):
(a) (5 points) Determine whether $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt[3]{k}}$ converges absolutely, converges conditionally, or diverges.

$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt[3]{k}}$ cannot converge absolutely since $\displaystyle\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{\sqrt[3]{k}} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$ is a p-series with $p < 1$.

However, since $\displaystyle\lim_{k\to\infty} \frac{1}{\sqrt[3]{k}} = 0$ and $\displaystyle\frac{d}{dx}x^{-\frac{1}{3}} = -\frac{1}{3}x^{-\frac{4}{3}} < 0$ for $x \geq 1$, the terms of the series decrease to 0 and $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt[3]{k}}$ converges by the Alternating Series Test.

Therefore, $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt[3]{k}}$ converges conditionally.

(b) (5 points) Find an upper bound on the remainder when the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$ is approximated by its first 3 terms.

The remainder satisfies $R_n < a_{n+1} \implies R_3 < a_4 = \dfrac{1}{64}$. Therefore $\dfrac{1}{64}$ is an upper bound for the remainder of the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$ when it is approximated by its first 3 terms.

Problem 9 (10 points):

(a) (5 points) Determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+1}$ converges or diverges.

$\displaystyle\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0$, so the series diverges by the Divergence Test.

(b) (5 points) Determine whether $\displaystyle\sum_{k=1}^{\infty} \sin k$ converges or diverges.

$\displaystyle\lim_{k\to\infty} \sin k$ does not exist, so the series diverges by the Divergence Test.

Problem 10 (10 points): Determine all values of $x \geq 0$ for which $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$ converges.

We use the Ratio Test: $\displaystyle r = \lim_{k\to\infty} \frac{a_{n+1}}{a_n} = \lim_{k\to\infty} \frac{\frac{x^{2(k+1)}}{(k+1)^2}}{\frac{x^{2k}}{k^2}} = \lim_{k\to\infty} \frac{x^2k^2}{(k+1^2)} = x^2$

For $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$ to converge we need $r < 1 \implies x^2 < 1$. Checking the endpoint $x = 1$ gives us $\sum_{k=1}^{\infty} \frac{1}{k^2}$ which is a convergent p-series.

Therefore, $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$ converges for x such that $0 \leq x \leq 1$

Extra Credit (5 points) Given that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, determine the value of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$.

$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{6} + ...$ and $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{6} + ...$

so subtracting yields $\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{16}\right) + ...$

$$= 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = 2\left(\frac{1}{4}\right)\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2}\frac{\pi^2}{6} = \frac{\pi^2}{12}$$

Therefore, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{6} - \frac{\pi^2}{12} = \frac{\pi^2}{12}$