This is a two hour exam. Show all work for full credit. No electronic devices or formula sheets are allowed.

Problem 1 (10 points): Evaluate $\int \sin^2 x \cos^2 x \ dx$.

$$\int \sin^2 x \cos^2 x \ dx = \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right) dx$$

$$= \frac{1}{4}\int (1-\cos^2(2x))dx = \frac{1}{4}\int \left(1-\left(\frac{1}{2}+\cos(4x)\right)\right) dx$$
$$= \frac{x}{8} - \frac{\sin(4x)}{32} + C$$

Problem 2 (10 points): Evaluate $\int \tan x \sec^4 x \ dx$.

$$\int \tan x \sec^4 x \ dx = \int \tan x(1+\tan^2 x)\sec^2 x \ dx$$

Let $u = \tan x$ so that $du = \sec^2 x dx$ Then we have

$$\int u(1+u^2)du = \int (u+u^3)du = \frac{u^2}{2} + \frac{u^4}{4} + C$$

$$= \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$$

Problem 3 (10 points):

(a) (5 points) Evaluate $\int_0^{\ln 2} xe^{2x} dx$.

Let $u = x, dv = e^{2x}dx$ so that $du = dx, v = \frac{e^{2x}}{2}$.

Then we have $\frac{xe^{2x}}{2}\Big|_0^{\ln 2} - \int_0^{\ln 2} \frac{e^{2x}}{2} dx = \frac{xe^{2x}}{2}\Big|_0^{\ln 2} - \frac{1}{4}e^{2x}\Big|_0^{\ln 2} = \ln 2 \frac{e^{2\ln 2}}{2} - \frac{1}{4}e^{\ln 4} + \frac{1}{4} = 2\ln 2 - \frac{3}{4}$

(b) (5 points) Evaluate $\int e^x \sin x dx$.

Let $u = \sin x, dv = e^x dx$ so that $du = \cos x dx, v = e^x$.

Then we have $\sin x e^x - \int e^x \cos x dx$. Let $u = \cos x, dv = e^x dx$ so that $du = -\sin x dx, v = e^x$.

Then we have $\int e^x \sin x dx = \sin x e^x - \cos x e^x - \int e^x \sin x dx$

$$\implies \int e^x \sin x dx = \frac{\sin x e^x - \cos x e^x}{2} + C$$

Problem 4 (10 points):

Let m and n be integers with $m \neq n$. Use the fact that

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right]$$

to show that $\int_0^\pi \cos(mx)\cos(nx)dx = 0$. This relation is useful for generating Fourier series.

$$\int_0^\pi \cos(mx)\cos(nx)dx = \int_0^\pi \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right]$$

Let $u = (m-n)x, v = (m+n)x$

so that $du = (m-n)dx, dv = (m+n)dx, u(0) = v(0) = 0, u(\pi) = (m-n)\pi, v(\pi) = (m+n)\pi$. Then we have

$$\frac{1}{2}\left[\frac{1}{m-n}\int_0^{(m-n)\pi} \cos u du + \frac{1}{m+n}\int_0^{(m+n)\pi} \cos v dv\right]$$

$$= \frac{1}{2}\left[\frac{1}{m-n}\sin u\Big|_0^{(m-n)\pi} + \frac{1}{m+n}\sin v\Big|_0^{(m+n)\pi}\right] = 0$$

Problem 5(10 points): Find the partial fraction expansion of $\frac{z+1}{z^2(z^2+4)}$.

$$\frac{z+1}{z^2(z^2+4)} = \frac{A}{z} + \frac{B}{z^2} + \frac{Cz+D}{z^2+4}$$

We have $z+1 = Az(z^2+4) + B(z^2+4) + (Cz+D)z^2 = (A+C)z^3 + (B+D)z^2 + 4Az + 4B$

so that $4B = 1 \implies B = \frac{1}{4}, 4A = 1 \implies A = \frac{1}{4}, A + C = 0 \implies C = -\frac{1}{4}, B + D = 0 \implies D = -\frac{1}{4}$.

Thus $\frac{z+1}{z^2(z^2+4)} = \frac{1}{4}\left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4}\right)$

Problem 6 (10 points): Evaluate $\int_1^4 \frac{dx}{x^2-2x+5}$.

Completing the square yields $\int_1^4 \frac{dx}{x^2-2x+5} = \int_1^4 \frac{dx}{(x-1)^2+4} = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right)\Big|_1^4$

$= \frac{1}{2}\arctan\left(\frac{3}{2}\right) - \frac{1}{2}\arctan(0) = \frac{1}{2}\arctan\left(\frac{3}{2}\right)$

Problem 7 (10 points): Evaluate $\int \frac{dy}{y^2\sqrt{9y^2-81}}$ for $y > 3$.

$\int \frac{dy}{y^2\sqrt{9y^2-81}} = \frac{1}{3}\int \frac{dy}{y^2\sqrt{y^2-9}}$.

Let $y = 3\sec\theta$ so that $y^2 = 9\sec^2\theta, dy = 3\sec\theta\tan\theta d\theta, \sqrt{y^2-9} = 3\tan\theta$.

Then we have $\frac{1}{3}\int \frac{d\theta}{9\sec\theta} = \frac{1}{27}\sin\theta + C = \frac{1}{27}\left(\frac{\sqrt{y^2-9}}{y}\right) + C$

Problem 8(10 points): Evaluate $\int \frac{2x^3+x^2-6x+7}{x^2+x-6}dx$.

Long division shows that $\frac{2x^3+x^2-6x+7}{x^2+x-6} = 2x - 1 + \frac{7x+1}{(x-3)(x+2)}$.

We write $\frac{7x+1}{(x-3)(x+2)} = \frac{A}{x+3} + \frac{B}{x-2} \implies 7x+1 = A(x-2)+B(x+3) \implies A = 4, B = 3$.

Thus $\int \frac{2x^3+x^2-6x+7}{x^2+x-6} dx = \int \left(2x-1+\frac{4}{x+3}+\frac{3}{x-2}\right) dx$

$= x^2 - x + 4\ln|x+3| + 3\ln|x-2| + C$

Problem 9 (10 points): Given a function $f(t)$, the *Laplace transform* is a new function $F(s)$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

where we assume s is a positive real number. Find the Laplace transform of $f(t) = t$.

$F(s) = \lim_{b\to\infty} \int_0^b e^{-st} t dt$. Let $u = t, dv = e^{-st}dt$ so that $du = dt, v = -\frac{e^{-st}}{s}$.

Then we have $F(s) = \lim_{b\to\infty} \left(-\frac{te^{-st}}{s}\right|_0^b + \lim_{b\to\infty} \int_0^b \frac{e^{-st}}{s} dt = \lim_{b\to\infty} \left(-\frac{te^{-st}}{s}\right|_0^b - \lim_{b\to\infty} \left(\frac{e^{-st}}{s^2}\right|_0^b = \frac{1}{s^2}$

Problem 10 (10 points):
(a) (5 points) Determine whether $\int_{-\infty}^{\infty} e^{-|x|} dx$ converges or diverges.

$\int_{-\infty}^{\infty} e^{-|x|} dx = \lim_{a\to-\infty} \int_a^0 e^x dx + \lim_{b\to\infty} \int_0^b e^{-x} dx = \lim_{a\to-\infty} \left(e^x\right|_a^0 + \lim_{b\to\infty} \left(e^{-x}\right|_0^b = 2$

Therefore $\int_{-\infty}^{\infty} e^{-|x|} dx$ converges.

(b) (5 points) Determine whether $\int_{-1}^{1} \frac{dx}{x^3}$ converges or diverges.

$\int_{-1}^{1} \frac{dx}{x^3} = \lim_{a\to 0^-} \int_{-1}^{a} + \lim_{b\to 0^+} \int_b^1 = \lim_{a\to 0^-} \left(-\frac{1}{2x^2}\right|_{-1}^{a} + \lim_{b\to 0^+} \left(-\frac{1}{2x^2}\right|_b^1 = -\infty + \infty$

Therefore $\int_{-1}^{1} \frac{dx}{x^3}$ diverges.

Extra Credit (10 points) Determine whether $\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$ converges or diverges.

$$\int_0^\infty \frac{dx}{\sqrt{x}+x^2} = \lim_{a\to 0^+}\int_a^1 \frac{dx}{\sqrt{x}+x^2} + \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}+x^2}.$$

We use the limit comparison test on each interval. Compare the integrand with $\dfrac{1}{\sqrt{x}}$ on $(0,1]$ and with $\dfrac{1}{x^2}$ on $(1,\infty)$.

We have $\displaystyle\lim_{x\to 0^+}\frac{\sqrt{x}+x^2}{\sqrt{x}} = 1$ and $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}+x^2}{x^2} = 1$.

Therefore, $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$ converges if and only if $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ both converge.

Now, $\displaystyle\lim_{a\to 0^+}\int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+}\left(2\sqrt{x}\Big|_a^1 = 2 \implies \int_0^1 \frac{1}{\sqrt{x}}\right.$ converges

and $\displaystyle\lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left(-\frac{1}{x}\Bigg|_1^b = 1 \implies \int_1^\infty \frac{1}{x^2}\right.$ converges.

Therefore $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$ converges.