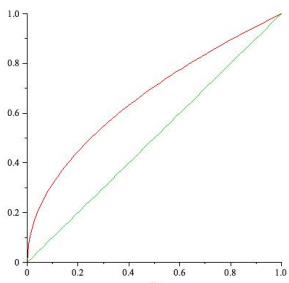


This is a two hour exam. Show all work for full credit. No electronic devices or formula sheets are allowed.

Problem 1 (10 points):

(a) (5 points) Find the volume of the solid generated by revolving the region bounded by the curves and lines  $y = x$ ,  $y = \sqrt{x}$  about the y-axis using the washer method.



$$V = \int_0^1 \pi (y^2 - (y^2)^2) dy = \pi \left( \frac{y^3}{3} - \frac{y^5}{5} \Big|_0^1 = \frac{2\pi}{15} \right)$$

(b) (5 points) Find the volume of the solid generated by revolving the region bounded by the curves and lines  $y = x$ ,  $y = \sqrt{x}$  about the y-axis using the shell method.

$$V = \int_0^1 2\pi x (\sqrt{x} - x) dy = 2\pi \int_0^1 \left( x^{\frac{3}{2}} - x^2 \right) dy = 2\pi \left( \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^3}{3} \Big|_0^1 = \frac{2\pi}{15} \right)$$

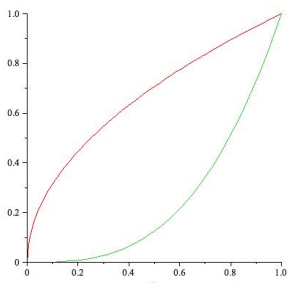
Problem 2 (10 points): Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, and initial position. Assume units of meters and seconds.

$$a(t) = e^{-2t}, v(0) = 10, s(0) = 20$$

$$v(t) = v(0) + \int_0^t a(x) dx = 10 + \int_0^t e^{-2x} dx = 10 - \frac{e^{-2t}}{2} + \frac{1}{2} = \frac{21}{2} - \frac{e^{-2t}}{2} \frac{m}{s}$$

$$s(t) = s(0) + \int_0^t v(x) dx = 20 + \int_0^t \left( \frac{21}{2} - \frac{e^{-2x}}{2} \right) dx = 20 + \frac{21t}{2} - \frac{1}{4} = \frac{79}{4} + \frac{21t}{2} + \frac{e^{-2t}}{4} m$$

Problem 3 (10 points): (a) (5 points) Find the area of the region bounded by  $y = \sqrt{x}$  and  $y = x^3$  using an integral with respect to x.



$$A = \int_0^1 (\sqrt{x} - x^3) dx = \left( \frac{2}{3} x^{\frac{3}{2}} - \frac{x^4}{4} \Big|_0^1 = \frac{5}{12} \right)$$

(b) (5 points) Find the area of the region bounded by  $y = \sqrt{x}$  and  $y = x^3$  using an integral with respect to  $y$ .

$$A = \int_0^1 (\sqrt[3]{y} - y^2) dy = \left( \frac{3}{4}y^{\frac{4}{3}} - \frac{y^3}{3} \right) \Big|_0^1 = \frac{5}{12}$$

Problem 4 (10 points): A quantity increases exponentially in time according to the function  $y(t) = y_0 e^{kt}$ . Find the time required for the quantity to increase  $p$ -fold.

$$py_0 = y_0 e^{kt} \implies p = e^{kt} \implies \ln p = kt \implies t = \frac{\ln p}{k}$$

Problem 5 (15 points):

(a) (5 points) Evaluate  $\frac{d}{dx}(x^x)$

$$\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x \ln x}) = e^{x \ln x} \left( \ln x + x \left( \frac{1}{x} \right) \right) = x^x (\ln x + 1)$$

(b) (5 points) Evaluate  $\int 3^{3x} dx$

$$\int 3^{3x} dx = \frac{3^{3x}}{3 \ln 3} + C$$

(c) (5 points) Use the definition of the natural logarithm to prove that  $\ln(xy) = \ln x + \ln y$

$$\ln(xy) = \int_1^{xy} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_x^{xy} \frac{1}{t} dt.$$

Let  $u = \frac{t}{x}$  so that  $du = \frac{1}{x} dx$ . Also, we have  $u(x) = 1, u(xy) = y$ .

$$\text{Then we have } \int_1^x \frac{1}{t} dt + \int_x^{xy} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_1^y \frac{1}{u} du = \ln x + \ln y$$

Problem 6 (10 points):

(a) (5 points) Find the mass of a thin bar with density  $|x - 1| \frac{kg}{m}$  for  $0 \leq x \leq 2$ .

$$m = \int_0^1 (1 - x) dx + \int_1^2 (x - 1) dx = \left( x - \frac{x^2}{2} \right) \Big|_0^1 + \left( \frac{x^2}{2} - x \right) \Big|_1^2 = 1 \text{ kg}$$

(b) (5 points) A linear spring can be stretched and held 0.5 m from its equilibrium position with a force of 20 N. Find the work done in compressing the spring 0.5 m from its equilibrium position.

We use Hooke's Law  $F(x) = kx$ :  $0.5k = 20 \implies k = 40 \frac{N}{m}$ .

$$\text{Now } W = \int_0^{0.5} F(x) dx = \int_0^{0.5} 40x dx = \left( \frac{40x^2}{2} \right) \Big|_0^{0.5} = 5 \text{ J}$$

Problem 7 (5 pts): Find a curve which passes through the point  $\left(1, \frac{1}{5}\right)$  and has an arc length

on the interval  $[3,6]$  given by  $\int_3^6 \sqrt{1+x^8} dx$ .

$$(y')^2 = x^8 \implies y' = \pm x^4 \implies y = \pm \frac{x^5}{5} + C \implies y = \frac{x^5}{5} \text{ or } y = -\frac{x^5}{5} + \frac{2}{5}$$

Problem 8 (10 points): Suppose the acceleration of an object moving along a line is given by  $a(t) = kv(t) \frac{m}{s^2}$ , where  $k$  is a positive constant and  $v$  is the object's velocity. Assume that the initial position and velocity are given by  $s(0) = 0$  m and  $v(0) = 20 \frac{m}{s}$

(a) (5 points) Use  $a(t) = v'(t)$  to find the velocity of the object as a function of time.

$$\begin{aligned} v'(t) = kv(t) &\implies \int \frac{v(t)}{v'(t)} dt = \int k dt \implies \ln v(t) = kt + C \implies v(t) = e^{kt+C} \\ &\implies v(t) = e^C e^{kt} \frac{m}{s}. \end{aligned}$$

$$\text{Now } v(0) = 20 \frac{m}{s} \implies v(t) = 20e^{kt} \frac{m}{s}.$$

(b) (5 points) Use  $v(t) = s'(t)$  to find the position of the object as a function of time.

$$s(t) = s(0) + \int v(t) dt = s(0) + \frac{20}{k} e^{kt} - \frac{20}{k} m.$$

$$\text{Now } s(0) = 0 \implies s(t) = \frac{20}{k} e^{kt} - \frac{20}{k} m.$$

Problem 9 (10 points): Find the length of the curve  $y = \frac{x^3}{6} + \frac{1}{2x}$  on the interval  $[1,2]$

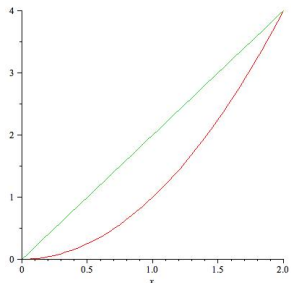
$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$(y')^2 = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$$

$$L = \int_1^2 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} dx = \int_1^2 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} dx = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} dx$$

$$= \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left(\frac{x^3}{6} - \frac{1}{2x}\right) \Big|_1^2 = \frac{17}{12}$$

Problem 10 (10 points): Let  $R$  be the region bounded by the curves  $y = x^2$  and  $y = 2x$ . Find the volume of the solid generated when  $R$  is revolved around the  $x$ -axis.



Using the washer method:

$$V = \int_0^2 \pi ((2x)^2 - (x^2)^2) dx = \int_0^2 \pi (4x^2 - x^4) dx = \pi \left( \frac{4x^3}{3} - \frac{x^5}{5} \right) \Big|_0^2 = \frac{64\pi}{15}$$

Using the shell method:

$$V = \int_0^4 2\pi y \left( \sqrt{y} - \frac{y}{2} \right) dy = 2\pi \int_0^4 \left( y^{\frac{3}{2}} - \frac{y^2}{2} \right) dy = 2\pi \left( \frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^3}{6} \right) \Big|_0^4 = \frac{64\pi}{15}$$