Exercises 1.1

1. (a) [BB]

p	q	$\neg q$	$(\neg q) \vee p$	$p \wedge ((\neg q) \vee p)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	F

(b)

p	q	$\neg p$	$(\neg p) \to q$	$p \wedge q$	$(p \wedge q) \vee ((\neg p) \to q)$
T	T	F	T	T	T
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	F	F

(c)

p	q	$q \vee p$	$p \wedge (q \vee p)$	$\neg (p \wedge (q \vee p))$	$\neg (p \wedge (q \vee p)) \leftrightarrow p$
T	T	T	T	F	F
T	F	T	T	F	F
F	T	T	F	T	F
F	F	F	F	T	F

(d) [BB]

p	q	r	$\neg q$	$p \vee (\neg q)$	$\neg (p \vee (\neg q))$	$\neg p$	$(\neg p) \vee r$	$(\neg (p \vee (\neg q))) \wedge ((\neg p) \vee r)$
T	T	T	F	T	F	F	T	F
T	F	T	T	T	F	F	T	F
F	T	T	F	F	T	T	T	T
F	F	T	T	T	F	T	T	F
T	T	F	F	T	F	F	F	F
T	F	F	T	T	F	F	F	F
F	T	F	F	F	T	T	T	T
F	F	F	T	T	F	T	T	F

(e)

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \vee r$	$(p \to (q \to r)) \to ((p \wedge q) \vee r)$
T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T
F	T	T	T	T	F	T	T
F	F	T	T	T	F	T	T
T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F
F	T	F	F	T	F	F	F
F	F	F	T	T	F	F	F

2. (a) If $p \to q$ is false, then necessarily p is true and q is false. (This is the only situation in which $p \to q$ is false.) We construct the relevant row of the truth table for $\big(p \wedge (\neg q)\big) \ \vee \ \big((\neg p) \to q\big)$.

p	q	$\neg q$	$p \wedge (\neg q)$	$\neg p$	$(\neg p) \to q$	$\big(p \wedge (\neg q)\big) \ \vee \ \big((\neg p) \to q\big)$
T	F	T	T	F	T	T

(b) [BB] There are three situations in which $p \to q$ is true. The question then is whether or not the truth value of $\big(p \wedge (\neg q)\big) \ \vee \ \big((\neg p) \to q\big)$ is the same in each of these cases. We construct a partial truth table.

p	q	$\neg q$	$p \wedge (\neg q)$	$\neg p$	$(\neg p) \to q$	$\big(p \wedge (\neg q)\big) \ \vee \ \big((\neg p) \to q\big)$
T	T	F	F	F	T	T
F	F	T	F	T	F	F

As shown, $(p \wedge (\neg q)) \vee ((\neg p) \to q)$ has different truth values on two occasions where $p \to q$ is

(b) [BB]

p	q	$\neg p$	$(\neg p) \wedge q$	$\neg q$	$p \vee (\neg q)$	$((\neg p) \wedge q) \wedge (p \vee (\neg q))$
T	T	F	F	F	T	F
T	F	F	F	T	T	F
F	T	T	T	F	F	F
F	F	T	F	T	T	F

The final column shows that $((\neg p) \wedge q) \wedge (p \vee (\neg q))$ is false for all values of p and q, so this statement is a contradiction.

6. (a)

p	q	$p \to q$	$q \to (p \to q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

Since $q \to (p \to q)$ is true for all values of p and q, this statement is a tautology.

(b)

p	q	$p \wedge q$	$\neg p$	$\neg q$	$((\neg p) \vee (\neg q))$	$(p \wedge q) \wedge [(\neg p) \vee (\neg q)]$
T	T	T	F	F	F	F
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

Since $(p \wedge q) \wedge ((\neg p) \vee (\neg q))$ is false for all values of p and q, this statement is a contradiction.

7. (a) [BB]

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$
T	T	T	T	T	T	T
T	F	T	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	T	T	T
T	T	F	T	F	F	F
T	F	F	F	T	F	F
F	T	F	T	F	F	T
F	F	F	T	T	T	T

$[(p \to q) \wedge (q \to r)] \to [p \to r]$
T
T
T
T
T
T
T
T

Since $[(p \to q) \wedge (q \to r)] \to [p \to r]$ is true for all values of p, q, and r, this statement is a tautology.

(b) [BB] If p implies q which, in turn, implies r, then certainly p implies r.

8. We must show that the given "or" statement can be both true and false. We construct truth tables for each part of the "or" and show that certain identical values for the variables make both parts T (so that the "or" is true) and other certain identical values for the variables make both parts F (so that the "or" is false).

p	r	s	$\neg r$	$\neg s$	$(\neg r) \to (\neg s)$	$p \vee [(\neg r) \to (\neg s)]$
T	T	T	F	F	T	T
F	F	T	T	F	F	F

p	q	r	s	t	$\neg t$	$(\neg t) \vee p$	$s \to [(\neg t) \vee p]$
T	T	T	T	T	F	T	T
F	F	F	T	T	F	F	F

$\neg q$	$(\neg q) \to r$	$[s \to ((\neg t) \vee p)] \vee [(\neg q) \to r]$
F	T	T
T	F	F

9. We are given that $\mathcal{A}$ is false for any values of its variables.

   (a) [BB] An implication $p \to q$ is false only if p is true and q is false. Since $\mathcal{A}$ is always false, $\mathcal{A} \to \mathcal{B}$ is always true. So it is a tautology.

   (b) An implication $p \to q$ is false only if p is true and q is false. Since $\mathcal{A}$ is false and the tautology $\mathcal{B}$ is true for any values of the variables they contain, $\mathcal{B} \to \mathcal{A}$ is always false. So it is a contradiction.

10. (a) The tables below show that when all three variables p, q and r are false, $p \to (q \to r)$ is true, whereas $(p \to q) \to r$ is false. Thus these statements have different truth tables and hence are not logically equivalent.

p	q	r	$q \to r$	$p \to (q \to r)$
F	F	F	T	T

p	q	r	$p \to q$	$(p \to q) \to r$
F	F	F	T	F

   (b) The compound statement is false.

11. (a) [BB]

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

   (b)

p	q	$\neg p$	$(\neg p) \wedge q$	$p \veebar ((\neg p) \wedge q)$	$(p \veebar ((\neg p) \wedge q)) \vee q$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	F	F	F

   (c) [BB]

p	q	$p \veebar q$	$p \vee q$	$(p \veebar q) \to (p \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T.

   The truth table shows that $(p \veebar q) \to (p \vee q)$ is true for all values of p and q, so it is a tautology.

   (d)

p	q	$p \veebar q$	$p \leftrightarrow q$	$\neg (p \leftrightarrow q)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

   Columns three and five are the same. So the truth values of $p \veebar q$ and $\neg (p \leftrightarrow q)$ are the same for all values of p and q. Thus these statements are logically equivalent.

Since $p \vee [\neg(p \wedge q)]$ is true for all values of p and q, this statement is a tautology.

T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(b) By one of the laws of DeMorgan, the negation is $(\neg p) \wedge (p \wedge q)$. By associativity, this is logically equivalent to $[(\neg p) \wedge p] \wedge q \iff \mathbf{0} \wedge q \iff \mathbf{0}$. So the negation is a contradiction.

3. (a) [BB] Using one of the laws of De Morgan and one distributive property, we obtain

$$\begin{aligned}[(p \wedge q) \vee (\neg((\neg p) \vee q))] &\iff [(p \wedge q) \vee (p \wedge (\neg q))] \\ &\iff [p \wedge (q \vee (\neg q))] \iff (p \wedge \mathbf{1}) \iff p.\end{aligned}$$

(b) The given compound statement is of the form $x \to (y \to z)$ which is equivalent to $x \to ((\neg y) \vee z) \iff (\neg x) \vee (\neg y) \vee z$. (By associativity, no further parentheses are required here.) So the given statement is equivalent to $(\neg(p \vee r)) \vee ((\neg q) \wedge r) \vee (p \vee r)$. By commutativity, this is $(\neg(p \vee r)) \vee (p \vee r) \vee ((\neg q) \wedge r) \iff \mathbf{1} \vee ((\neg q \wedge r) \iff \mathbf{1}$. The given statement is a tautology!

(c) Using associativity to avoid extra parentheses, the left side of the given statement is

$$\begin{aligned}[(p \to q) \vee (q \to r)] &\iff (\neg p) \vee q \vee (\neg q) \vee r \\ &\iff (\neg p) \vee \mathbf{1} \vee r \iff \mathbf{1}.\end{aligned}$$

The given statement is equivalent to $[\mathbf{1} \wedge (r \to s)]$ which is logically equivalent to $r \to s$.

4. (a) [BB]

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

(b)

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

5. (a) [BB] Distributivity gives $[(p \vee q) \wedge (\neg p)] \iff [(p \wedge (\neg p)) \vee (q \wedge (\neg p))] \iff [\mathbf{0} \vee ((\neg p) \wedge q)] \iff [(\neg p) \wedge q]$.

(b) We have $(p \to (q \to r)) \iff (p \to ((\neg q) \vee r)) \iff (\neg p) \vee (\neg q) \vee r$
$\iff (\neg p) \vee r \vee (\neg q) \iff \neg(p \wedge (\neg r)) \vee (\neg q) \iff (p \wedge (\neg r)) \to (\neg q)$.

(c) $(\neg(p \leftrightarrow q)) \iff (\neg((p \to q) \wedge (q \to p))) \iff (\neg(((\neg p) \vee q) \wedge ((\neg q) \vee p)))$

$$\begin{aligned}&\iff ((p \wedge (\neg q)) \vee (q \wedge (\neg p))) \\ &\iff ((p \wedge (\neg q)) \vee q) \wedge ((p \wedge (\neg q)) \vee (\neg p)) \\ &\iff ((p \vee q) \wedge ((\neg q) \vee q)) \wedge ((p \vee (\neg p)) \wedge ((\neg q) \vee (\neg p))) \\ &\iff ((p \vee q) \wedge \mathbf{1}) \wedge (\mathbf{1} \wedge ((\neg q) \vee (\neg p))) \\ &\iff (p \vee q) \wedge ((\neg q) \vee (\neg p)) \\ &\iff ((\neg p) \vee (\neg q)) \wedge (q \vee p) \\ &\iff (p \to (\neg q)) \wedge ((\neg q) \to p) \iff (p \leftrightarrow (\neg q)).\end{aligned}$$

(d) [BB] $\neg[(p \leftrightarrow q) \vee (p \wedge (\neg q))] \iff [\neg(p \leftrightarrow q) \wedge \neg(p \wedge (\neg q))]$
$\iff [(p \leftrightarrow (\neg q)) \wedge ((\neg p) \vee q)]$, using Exercise 5(c).

(e) This is an immediate application of absorption law **4(b)** with $p \wedge (\neg q)$ in place of p and $q \wedge (\neg r)$ in place of q.

(f) Using property 12 and associativity, $[p \to (q \vee r)] \iff [(\neg p) \vee q \vee r] \iff [\neg(p \wedge (\neg q))] \vee r$ (by De Morgan) $\iff [p \wedge (\neg q) \to r]$ using 12 again.

(g) $\neg(p \vee q) \vee [(\neg p) \wedge q] \iff [(\neg p) \wedge (\neg q)] \vee [(\neg p) \wedge q]$ (DeMorgan) $\iff (\neg p) \wedge [(\neg q) \vee q]$ (distributivity) $\iff (\neg p) \vee \mathbf{1} \iff \neg p$

6. $[(p \wedge (\neg q)) \to q] \iff [(\neg(p \wedge (\neg q))) \vee q] \iff [((\neg p) \vee q) \vee q] \iff [(\neg p) \vee q]$.
$[(p \wedge (\neg q)) \to (\neg p)] \iff [(\neg(p \wedge (\neg q))) \vee (\neg p)] \iff [(\neg p) \vee q \vee (\neg p)] \iff [(\neg p) \vee q]$.
So these are both logically equivalent to $(\neg p) \vee q$.

7. (a) We must show that $\mathcal{A} \vee \mathcal{C}$ and $\mathcal{B} \vee \mathcal{C}$ have the same truth tables, given that $\mathcal{A}$ and $\mathcal{B}$ have the same