

**MATH 125-001 Spring 2007 Exam 2**  
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**Student Name**\_\_\_\_\_

**Problem 1.** Prove that the two sets of real numbers,  $A = (0,10)$  and  $B = (1,2) \cup (3, 4)$ , have equal cardinality. Hint: use *Schröder-Bernstein theorem*.

**Problem 2.** Prove the following statement:

$$\forall a, b \in \mathbb{N}, \gcd(a, b) \operatorname{lcm}(a, b) = a b$$

Hint: use the Fundamental Theorem of Arithmetic (FTA).

- Problem 3.** Using the mathematical induction principle in the strong form, prove that
- (a) any natural number greater than 1 has a prime divisor,
  - (b) any natural number greater than 1 has a unique prime decomposition (FTA).

**Problem 4.** Using the generating function method, prove the formula:

$$1+x^{2^{n+1}} = (1+x)(1-x+x^2-x^3+\dots-x^{2^{n-1}}+x^{2^n})$$

**Problem 5.** Consider the Towers of Hanoi puzzle with  $n$  disks (Figure 1).



Figure 1. Initial configuration of the Towers of Hanoi puzzle.

The task is to move all disks to the right peg. The rules are the following. At each step you can move any top disk to any peg, except that you cannot put a larger disk over a smaller one.

Describe a recursive algorithm and calculate the number of steps that your algorithm would require to solve the puzzle.

**Extra credit problem**

(solving this problem is not a requirement and will not add to your score above 50 points)

Identify (circle) the invalid argument in the following “proof”.

Proposition. A bijection may not have an inverse.

*Proof.*

- (i) If a function  $f: \mathbb{R} \rightarrow \mathbb{R}^+$  has an inverse  $f^{-1}$ , then the plot of  $y = f^{-1}(x)$  is the mirror-reflection of the plot of  $y = f(x)$  with respect to the line  $y = x$ .
- (ii) It follows from (i) that if the two plots are continuous and intersect each other, then they must intersect on the line  $y = x$ .
- (iii) Consider  $f(x) = a^x$  with  $a = 1/16$ . This  $f: \mathbb{R} \rightarrow \mathbb{R}^+$  is a bijection and a continuous function. Suppose its inverse is  $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $g(x) = \log_a(x)$ . It is easy to verify that the plots of  $f$  and  $g$  intersect at  $x = 0.5, y = 0.25$ .
- (iv) It follows from (iii) that the point of intersection is not on the line  $y = x$ , while it follows from (ii) that it is on the line  $y = x$ . Therefore, we have a contradiction, and  $g$  is not an inverse of  $f$ .
- (v) On the other hand, one can show that any function  $h$  other than  $g$  is not an inverse of the given  $f$ , because in this case  $f \circ h$  is not the identity on  $\mathbb{R}^+$ .
- (vi) Therefore, it follows from (iv) and (v) that  $f$  has no inverse. The conclusion is that a bijection may not have an inverse.