# Networks and Flows

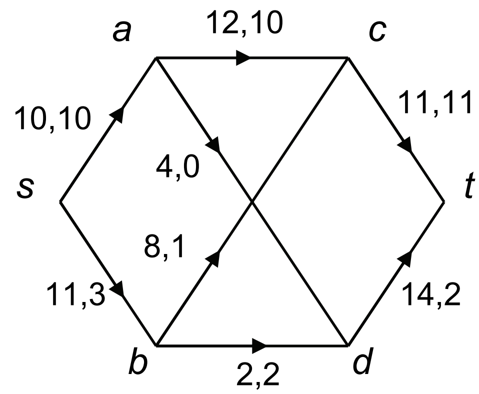


Figure 1

## Definitions:

Given a ***directed network*** (i.e., a digraph with integer weights: Fig. 1) with the weights of arcs understood as ***capacities*** $c_{ij}$ (e.g., a missing arc corresponds to a zero capacity), and with a set of vertices $V$ that includes two distinguished vertices called the ***source*** ($s$) and the ***sink*** ($t$), we define an **($s$, $t$) - flow** as a set of numbers $F = \{f_{uv}\}$ such that

$$0 \le f_{uv} \le c_{uv} \ \forall u,v \in V,$$
$$\sum_{v\in V} f_{uv} = \sum_{v\in V} f_{vu} \ \forall u \in V \setminus \{s,t\} \quad \text{(conservation of flow)},$$

and the ***value of flow*** as the net amount of flow leaving the source or, equivalently, the net amount of flow entering the sink:

$$\text{val}(F) = \sum_{v\in V}(f_{sv} - f_{vs}) = \sum_{v\in V}(f_{vt} - f_{tv}).$$

The equivalence of the two sums follows from the conservation of flow:

$$\sum_{v\in V}(f_{uv} - f_{vu}) = 0 \ \forall u \in V \setminus \{s,t\}, \quad \sum_{u,v\in V}(f_{uv} - f_{vu}) = 0,$$
$$\sum_{\substack{v\in V,\\ u\in V\setminus\{s,t\}}}(f_{uv} - f_{vu}) - \sum_{\substack{v\in V,\\ u\in V}}(f_{uv} - f_{vu}) = \sum_{v\in V}(f_{sv} - f_{vs}) - \sum_{v\in V}(f_{vt} - f_{tv}).$$

**<u>Definitions:</u>**

Given a directed network with the set of vertices $V$ including the source $s$ and the sink $t$, and with the capacities $c_{ij}$ , an **($s$, $t$) - cut** is a partition

$$V = S \cup T,\ S \cap T = \varnothing, s \in S, t \in T.$$

The ***capacity*** of an ($s$, $t$) - cut is

$$\text{cap}(S,T) = \sum_{\substack{u \in S,\\ v \in T}} c_{uv}.$$

**<u>Theorem:</u>**

In any directed network with the source $s$ and the sink $t$, the value of any ($s$, $t$) – flow never exceeds the capacity of any ($S$, $T$) – cut. More precisely,

$$\text{val}(F) = \sum_{\substack{u \in S,\\ v \in T}} \left(f_{uv} - f_{vu}\right) \le \sum_{\substack{u \in S,\\ v \in T}} c_{uv} = \text{cap}(S,T) \quad \forall F, (S,T).$$

*<u>Proof:</u>* the first equality follows from the conservation of flow.

**<u>Corollary:</u>** If in a directed network

$$\text{val}(F) = \text{cap}(S, T), \qquad (*)$$

then val ($F$) is the maximum, and cap ($S$, $T$) is the minimum over all possible flows and cuts in this network.

**<u>Max Flow – Min Cut Theorem</u>** (Ford and Fulkerson): for any directed network with a source and a sink, the equality (*) can be satisfied, so the ***maximal flow*** value is the ***minimal cut*** capacity.

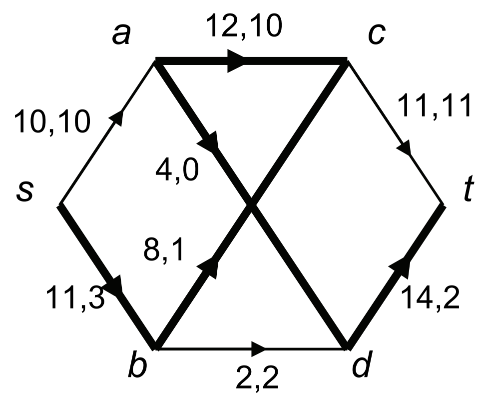
**Definitions:**

A ***unit flow***: $f = 1$ along a path from $s$ to $t$, and 0 otherwise.

A ***saturated arc***: $c = f$.
Example: in Figure 1 every path from $s$ to $t$ has a saturated arc.

A ***chain*** (Fig. 2) is a walk with distinct directed arcs that can be followed in either direction. A ***forward arc*** is followed in its assigned direction, and a ***backward arc*** is followed in the opposite direction.

A chain from source to sink is ***flow-augmenting*** iff for its every forward arc $f < c$ and for its every backward arc $f > 0$.
<u>Example:</u> chain *sbcadt* in Fig. 2.



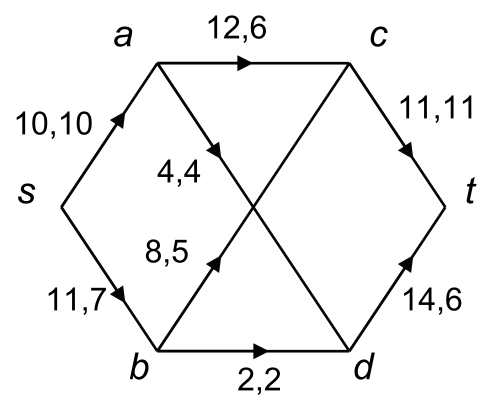




Figure 2. A maximum flow achieved by eliminating the slack of the chain *sbcadt*.

A flow-augmenting chain from source to sink can be used to increase the value of flow by increasing $f$ on every forward arc and decreasing $f$ on every backward arc by the same amount. The maximal increase of $f$ for a given chain is called the ***slack*** of a chain. <u>Example:</u> chain *sbcadt* in Fig. 2, top has a slack 4.

<u>*Proof of the theorem:*</u> Construct a flow such that there are no flow-augmenting chains from source to sink. Select a cut $(S, T)$ where $S$ includes all vertices reachable from the source by flow-augmenting chains. Then for every $u \in S$, $v \in T$, we have $f_{uv} = c_{uv}$ and $f_{vu} = 0$ (otherwise we could extend $S$). Therefore, the value of the flow is equal to the capacity of the cut. Now use the corollary.