Planar Graphs

Definitions:

A ***planar graph*** is a graph that can be drawn on a plane without self-intersections.

A ***plane graph*** is a graph that is drawn on a plane without self-intersections: i.e., it is an embedding of a graph into a plane.

A given plane graph partitions the plane into several regions, including the ***exterior*** and ***facets*** (connected regions of the plane surrounded by edges of the graph). It follows that in a plane graph:

- every region has a boundary that consists of edges of the graph;
- every edge belongs to a boundary of some region;
- if an edge belongs to a cycle in the graph, then it separates two regions;
- every edge can be drawn as a straight line segment (proof by conformal equivalence).

A graph is planar iff it can be drawn on a sphere without self-intersections. This statement follows from consideration of a mapping of a plane onto a sphere, such that the exterior of the plane graph becomes one of the facets on the sphere.

Then it follows that any ***polyhedron*** defines a symmetric planar graph: all polyhedra have the topology of a sphere.

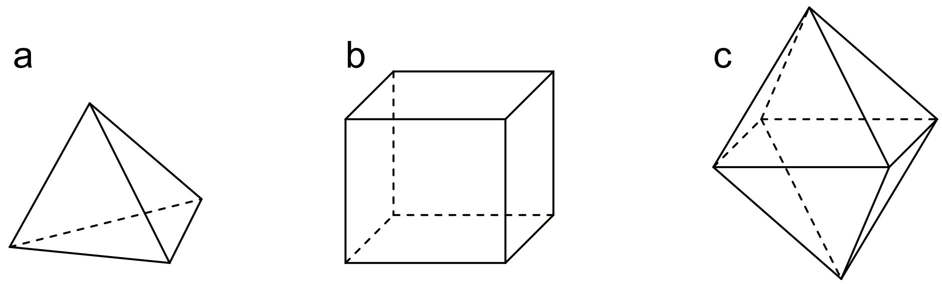


Figure 2. Three out of five regular convex polyhedra (Platonic solids): tetrahedron (a), cube (b), and octahedron (c). Vertices and edges form planar graphs.

An Introduction to Surface Topology

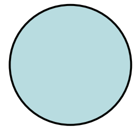
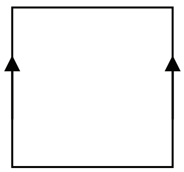
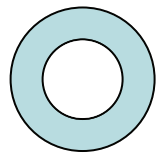
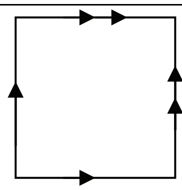
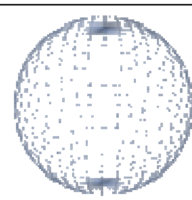
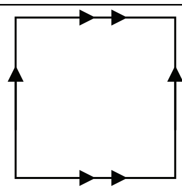
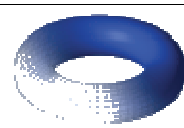
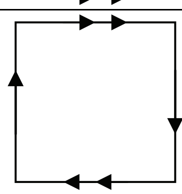
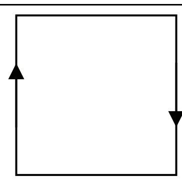
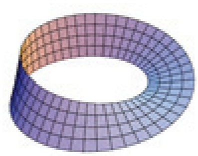
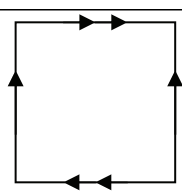
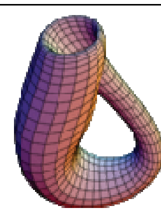
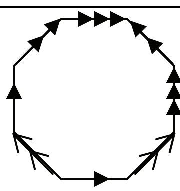
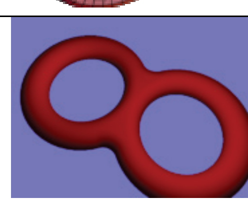
Definitions:

Intuitively, you know what a ***surface*** is. For example, you may think of a soap bubble, or a plastic bag, or a sheet of paper that you can cut with scissors and then glue in many different ways. The following material provides you with a shortcut through highly abstract concepts that cannot be rigorously introduced in this course.

Here we will be interested in the ***topology*** of surfaces, therefore, we will make no difference between two surfaces, if they can be continuously transformed into each other. Example: surfaces of a coffee mug and a donut have one and the same topology (a torus).
http://upload.wikimedia.org/wikipedia/commons/2/26/Mug_and_Torus_morph.gif

From this point of view, a compact surface can be defined by its ***fundamental polygon***, in which we need to specify pairs of edges that are "glued" to each other and their relative orientation with respect to each other (in figures below, arrows should match).

Table 1. Examples of compact connected surfaces (✓: orientable)

Name	Symbol	Fundamental polygon	3d immersion	Euler char. χ	Genus g or k
disk	D		 ✓	1	0
open annulus			 ✓	0	0
sphere	S^2		 ✓	2	0
torus	$\mathbb{T}^2$		 ✓	0	1
projective plane	$\mathbb{R}P^2$			1	1
Möbius strip				0	1
Klein bottle				0	2
double torus	$\mathbb{T}_2$		 ✓	-2	2
n-torus	$\mathbb{T}_n$		✓	2-2n	n

Definitions:

Recall that a ***triangulation*** of a surface is a partition of it by triangles (3-cycles) into facets.

More generally, ***polygonization*** (also called ***triangulation***) of a surface is partitioning it by n-cycles into planar domains – facets, each surrounded by edges of a cycle.

Euler's formula for Euler's characteristic χ of a surface defines the genus g of a surface:

Given a triangulation of an (1) orientable or (2) non-orientable surface that has c connected components and b open boundaries, the Euler's characteristic is given by the Euler formula:

$$\chi = v - e + f = 2c - 2g - b, \qquad (1)$$

$$\chi = v - e + f = 2c - g - b, \qquad (2)$$

where f is the number of facets, e is the number of edges, and v is the number of vertices in the triangulation. In particular, for a convex polyhedron (that can be drawn on a sphere) we have

$$\chi = v - e + f = 2. \qquad (3)$$

Therefore, the genus can be computed directly from the fundamental polygon (Table 1). For any open planar domain or a sphere $g = 0$, in this case (1) and (3) can be proven as theorems.

Theorem: $e \le 3v - 6$ for any planar graph with $v \ge 3$.

Corollaries: $K_{3,3}$ and K_5 are not planar.

Every planar graph contains at least one vertex of degree ≤ 5.

Definition:

Two graphs are ***homeomorphic*** iff they can be obtained from each other by adding / deleting vertices of degree 2 on existing edges.

E.g., any two cycles are homeomorphic.

Theorem: (***Kuratowski***) A graph is planar iff it has no subgraph homeomorphic to $K_{3,3}$ or K_5 .

Coloring Graphs

Definitions:

The ***dual graph*** of a given plane graph is the graph in which vertices represent facets (including the exterior) of the original graph, and each edge of the original graph is "flipped" in order to connect the two neighboring facets of the original graph.

Therefore, facets of a dual plane graph correspond to vertices of the original graph. Hence the graph duality relation is symmetric.

An n-***coloring*** of a graph is an assignment of n colors to vertices, such that every two adjacent vertices have different colors.

The ***chromatic number*** χ of a graph is the minimal n for which n-coloring exists. For example, $\chi(K_n) = n$, $\chi(K_{m,n}) = 2$.

The four-color theorem (Appel et al., 1977):
for any planar graph $\chi \leq 4$.

Coloring of a torus: for any connected graph drawn on a torus without self-intersections (analog of a planar graph), $\chi \leq 7$.