Paths and Circuits: Applications (continued)

Definitions

A ***Tournament*** is a digraph with exactly one arc connecting any two vertices.

The ***score*** in a tournament is the outdegree, and the ***score sequence*** is the outdegree sequence (i.e., the list of outdegrees in a non-decreasing order).

Theorem (Rédei): every tournament has a Hamiltonian path.

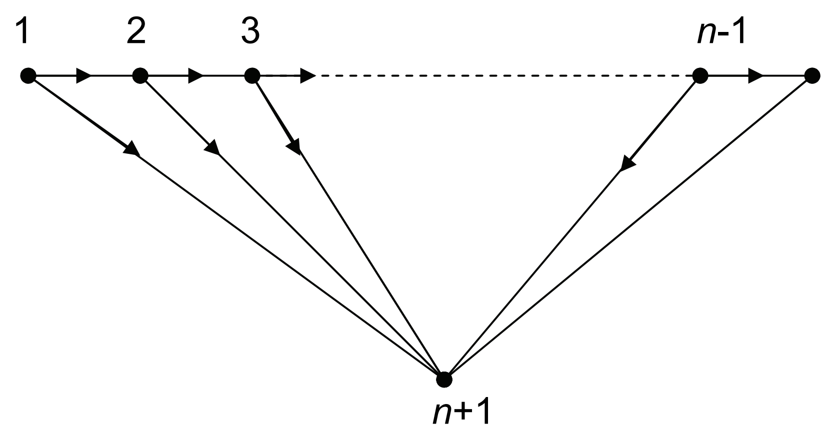

Figure 1

Proof: The statement is trivial for tournaments with $n = 1$ and $n = 2$ vertices. Suppose it is true for a tournament with $n > 1$ vertices (the induction hypothesis). Consider a tournament with $n+1$ vertices (Figure 1). Then, by the induction hypothesis, any its maximal subgraph of n vertices (necessarily a tournament) has a Hamiltonian path. Without any loss of generality, suppose that the vertices are numbered 1 through n in the order of their appearance in a Hamiltonian path of a selected subgraph of n vertices. Then the vertex $n+1$ has connections with all those n vertices. There are three possibilities: (i) all these n arcs (Figure 1) are directed toward

n+1, in which case a Hamiltonian path for the entire graph is (1, .., n, n+1); (ii) all n arcs are directed out of n+1, in which case a Hamiltonian path for the entire graph is (n+1, 1, .., n); and (iii) the set of n arcs is a mixture of inbound and outbound arcs, in which case there are only two possibilities: (a) one of the above two solutions works, or (b) there is an inbound arc followed by an outbound arc: i.e., an arc (i, n+1) followed by an arc (n+1, i+1) with $1 \le i < n$, and in this case a Hamiltonian path is obtained from (1, …, n) by reconnecting (i, i+1) as (i, n+1), (n+1, i+1). Therefore, assuming the induction hypothesis for n, we have established that the Hamiltonian path exists for a tournament of n+1 vertices. Therefore, by the induction principle, the theorem holds.

Definition

A tournament $T = (\{ v \}, E)$ is ***transitive*** iff
$uv \in E \wedge vw \in E \rightarrow uw \in E.$

Theorem: The following is equivalent.

1. T has a unique Hamiltonian path.

2. T is transitive.

3. Every player (i.e., vertex) in T has a different score.

Proof of 1→2.

The statement is easy to verify for tournaments with n = 1, 2, 3 and 4 vertices by consideration of all possibilities. Now we will prove that the statement holds for n = 5: i.e., that if a tournament T of 5 vertices has a unique Hamiltonian path, then it is transitive. As in the previous proof, without loss of generality, suppose that vertices are labeled in the order of their appearance in the unique Hamiltonian path: 12345 (Figure 2). If the tournament is transitive, then all arcs *i, j* with $i < j$ must be present in it. Suppose that this is not true: i.e., at least one of the arcs that are not included in the Hamiltonian path is "inverted". It cannot be the arc connecting the two endpoints, because then an alternative Hamiltonian path would be 51234. We consider three other candidates for inverted arcs: 31, 42 and 52, in all possible combinations. Suppose that we have an arc 31, then an alternative Hamiltonian path would be either 31245 or 42315, depending on whether 24 is inverted or not. Next, suppose that we have 42 and 13, then an alternative Hamiltonian path would be either 13425 or 15234, depending on whether 25 or 52 is in *E*. Finally, if we have 52, 13 and 24, then the last example of an alternative Hamiltonian path, 15234, still works. Now we do not need to consider candidates 41 and 53, due to the symmetry of the problem (consider inverting all orientations and numbers simultaneously). Therefore, the statement holds for $n = 5$.

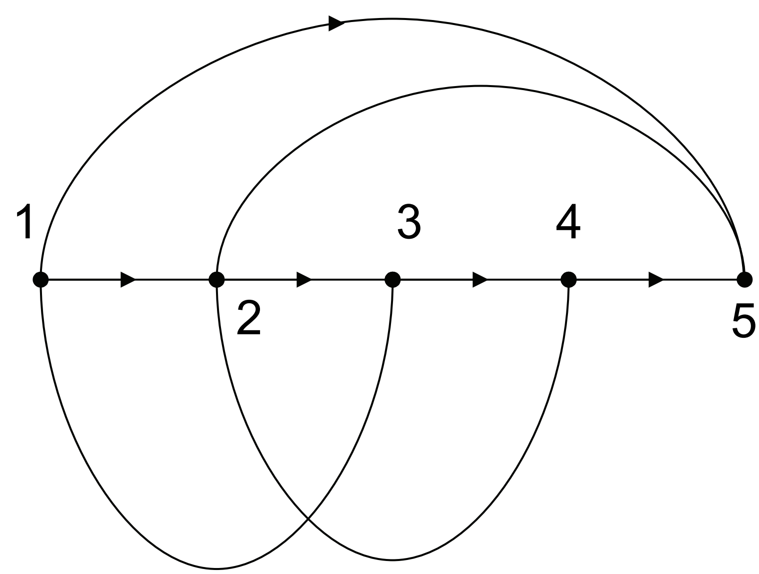

Figure 2.

How do we go about a general proof? We map it onto the proof that we just constructed. In this case we take 1, 2, 3, 4, 5 as labels rather than absolute positions of vertices. We assume that 42 is the largest among the leftmost inverted arcs, and that it does not include any endpoints (see below as to what if it does). We also assume, without loss of generality, that vertices 1 and 2, 2 and 3, as well as 4 and 5 are next to each other, while there may be additional vertices between 3 and 4 and also to the left of 1 and/or to the right of 5. Given these assumptions, the arcs 13 and 25 are not inverted, and the path [..13..425..] is Hamiltonian. Now suppose that the inverted arc includes an endpoint. We represent this situation by deleting either 1 or 5 from the graph in Figure 2, and the alternative Hamiltonian path becomes [3..425..] or [..13..4], respectively. As explained above, an inverted arc cannot include both endpoints without entailing another Hamiltonian path. Therefore, there is an alternative Hamiltonian path whenever there is an inverted arc not included in it. This completes the proof of the assertion $1 \to 2$.

The proof of $2 \to 1$, $1 \to 3$ and $2 \to 3$ is straightforward.