Induction and Recursion

Principle of mathematical induction (weak form):

$$\frac{P(n_0),\ n_0 \in \mathbb{Z} \qquad P(n) \to \mathrm{P}(n+1)\ \forall\, n \in \mathbb{Z},\ n \geq n_0}{P(n)\ \forall\, n \in \mathbb{Z},\ n \geq n_0}$$

$P(n)$ in the second line is called the induction hypothesis. Can prove this using the Well-Ordering Principle.

Principle of mathematical induction (strong form):

$$\frac{P(n_0),\ n_0 \in \mathbb{Z} \qquad \forall\, n \in \mathbb{Z},\ n \geq n_0\ \ (\ P(k)\ \forall\, k \in \{\ n_0 \ldots n\}\ ) \to \mathrm{P}(n+1)}{P(n)\ \forall\, n \in \mathbb{Z},\ n \geq n_0}$$

Follows from the weak form.
Example: prove that $\forall\, n \in \mathbb{N},\ 8 \mid (3^{2n}-1)$

Definition of the factorial:

$0!=1, \quad n! = n(n-1)(n-2)\ldots 1.$

Stirling's approximation: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Definition: A set A of integers is called an ***ideal*** iff
$0 \in A$,
$a \in A \to (-a) \in A$,
$a, b \in A \ \to\ a + b \in A$ prove that $n\mathbb{Z}$ is an ideal

Recursive definitions

Example:
$0!=1$, $\forall\, n \in \mathbb{N}$, $n! = (n-1)!\, n$ - check Stirling's approximation!

Prove that $n! \geq 3^{n-2}$

Definition:
In this chapter, we define a ***sequence*** as a function $f\colon \mathbb{N} \to \mathbb{R}$.

A sequence can be defined recursively
by ***initial condition*** and ***recursive relation***

example: $a_1=2$, $a_2=3$, $a_{k+1} = 2a_k + a_{k-1}\ \forall\, k>2$

arithmetic sequence: $a_1 = a$, $a_{k+1} = a_k + d$

d is ***common difference***

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

geometric sequence: $a_1 = a$, $a_{k+1} = ra_k$

r is ***common difference***

$$S_n = \frac{a\left(1-r^k\right)}{1-r}$$

Fibonacci sequence: $f_1 = 1$, $f_2 = 1$, $f_{k+1} = f_k + f_{k-1}$

Pseudorandom numbers: $x_{k+1} = (ax_k + b) \pmod{n}$

Next: Solving recurrence relations of the form

$$a_n = ra_{n-1} + sa_{n-2} + f(n)$$

using characteristic polynomials

Theorem: given a ***homogeneous*** recurrence relation

$a_n - ra_{n-1} - sa_{n-2} = 0,$

the solution is

$$a_n = \begin{cases} c_1 x_1^n + c_2 x_2^n & \text{if} \quad x_1 \neq x_2, \\ c_1 x^n + c_2 n x^n & x_1 = x_2 = x, \end{cases} \qquad (*)$$
$$x_1, x_2 \text{ are roots of } x^2 - rx - s = 0$$

In this case $x^2 - rx - s$ is called ***characteristic polynomial***.

The term comes from linear algebra:

$$V_1 = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}, \quad V_n = \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}, \quad A = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}, \qquad V_n = AV_{n-1}$$

Looking for a particular homogeneous solution: $V_n = xV_{n-1}$

$$V_n = x^n \xi, \quad A\xi = x\xi \quad \rightarrow \quad \det(A - x) = 0 \quad \rightarrow \quad x^2 - rx - s = 0,$$

$\xi = \begin{pmatrix} x \\ 1 \end{pmatrix}$. General homogeneous solution is the linear combination

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Theorem: The general solution to the recurrence relation

$$a_n - ra_{n-1} - sa_{n-2} = f(n)$$

is the sum of a particular solution and the general solution (*) to the homogeneous recurrence relation.

Definition:

Generating function of a sequence a_0, a_1, a_2, … is $a_0+a_1x+a_2x^2+\ldots$

Examples

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots$$

Example: construct generating function for Fibonacci sequence

$a_0 = 1$, $a_1 = 1$, $a_{k+1} = a_k + a_{k-1}$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$xf(x) = \quad a_0 x + a_1 x^2 + a_2 x^3 + \ldots$$

$$x^2 f(x) = \quad a_0 x^2 + a_1 x^3 + \ldots$$

$$(1 - x - x^2) f(x) = a_0 + (a_1 - a_0)x + \ldots + (a_{k+1} - a_k - a_{k-1})x^{k+1} + \ldots$$

$$= 1$$

$$\rightarrow \quad f(x) = \frac{1}{1 - x - x^2}$$

Now we can in principle calculate the coefficients.

Towers of Hanoi

Problem: construct a recursive definition of the algorithm that solves the puzzle for any number of disks

and compute the number of moves