Lecture 2: Sets, Relations and Functions

A ***set*** S is a collection of distinct objects $x_1, x_2, \ldots$ called ***elements*** (or ***members***) of the set. This is written as follows: $S = \{x_1, x_2, \ldots\}$, and the expression $x_1 \in S$ says that x_1 is a member of S (***belongs*** to S). The notion of a set does not allow for multiple instances (repetitions) of the same element in the set, while the notion of a ***sequence*** (an ordered collection) does. The notion of a set does not relate to the notion of continuity, while at the same time it provides a basis for all concepts of continuous data structures (which are not a topic here). As far as the set theory per se is concerned, the nature of elements is irrelevant: all that matters is how many elements are and whether two elements x and y are one and the same element ($x = y$) or not ($x \neq y$). For example, a set may contain another set (or even itself) as its element. A set can be empty (denoted by $\varnothing$), finite or infinite. The number of elements in a set S is called the ***cardinality*** of S, written as $|S|$.

Examples of sets are: natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$. A finite set can be defined by an explicit list of all its elements, for example: $\{1, 2, 3\}$ and $\{2\}$ are sets. However, $\{1, 2, 2, 3\}$ is not a set, because the list contains two instances of one and the same element. Alternatively, a set can be defined by a formula that specifies its elements, for example: $\{x : x \in \mathbb{R}, x > 2\}$.

Let A and B be two sets. They are ***equal***, $A = B$, iff they contain the same elements or both are empty. A is a ***subset*** of B (contained in B, $A \subseteq B$), iff every element of A is an element of B. In this case B is a ***superset*** of A ($B \supseteq A$). We say that A is a ***proper subset*** of B ($A \subset B$, or equivalently B is a proper superset of A, $B \supset A$) iff $A \subseteq B$ and $A \neq B$. Negations are expressed by slashing the corresponding symbols: for example, $\not\subset$, $\notin$.
The following is true for any sets A, B:

$$A \subseteq A$$
$$\varnothing \subseteq A$$
$$(A = B) \leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$$

The set of all subsets of A is called the ***power set*** of A, denoted $\mathcal{P}(A)$. The ***union*** $A \cup B$ is the set of all elements that appear in any of the two sets. The ***intersection*** $A \cap B$ is the set of all elements that appear in both sets. The ***set difference*** $A \backslash B$ (or $A\text{-}B$) is the set of those elements of A that are not in B. An intuition into these notions is given by the ***Venn diagram*** (page 44 in the textbook). Finally, ***symmetric set difference*** (denoted Δ or $\oplus$) is defined as $\mathrm{A} \oplus \mathrm{B} = (A \cup B) \setminus (A \cap B)$. ***Cartesian product*** (***direct product***) of sets A and B (denoted in this textbook as $\mathrm{A} \times \mathrm{B}$) is defined as the set of all ordered pairs, in each of which the first element belongs to A and the second element belongs to B. These elements are called ***coordinates*** of the ordered pair. Thus, an ***ordered pair*** is a sequence of two elements. An ***ordered n-tuple*** is a sequence of n elements. If X is a set, then X^n (the n-th ***power*** of X (not to be confused with the power set of X) is the set of all ordered n-tuples over X ("over X" here means "composed from elements of X").

Any given problem or context of consideration of some set A usually suggests the largest possible set in which A may be a subset; it is called the ***universal set*** U of a given context. Then the ***complement*** of A is understood as $A^c = U \backslash A$. It follows immediately that $(A^c)^c = A$, $U^c = \varnothing$, $\varnothing^c = U$.

From the point of view of associativity and distributivity, there is a direct analogy between $\cup$ and +, $\cap$ and $\cdot$, \ and -. The analogy is even deeper between $\cup$ and $\vee$, $\cap$ and $\wedge$, \ and $\wedge\neg$: in fact, logical operations that we studied in the previous lecture can be interpreted as set operations. In this case $\varnothing$ corresponds to contradiction, and U corresponds to tautology. This observation automatically extends all familiar equivalence relations (De Morgan's laws, etc.) as well as other facts about propositions to the current topic of sets.

Exercise (not for credit): construct a translation table connecting symbols and facts of propositional logic to symbols and facts of the set theory.

(to be continued)