MATH 125-001 Spring 2007 Final Exam
Instructor: Dr. Alexei V. Samsonovich
05/09/2007

Student Name___________________________________

Clearly state your answers and provide minimal explanation of your reasoning to them relevant to the questions. If you need more space, write on the back; do not turn in your scrap paper. You do not need to solve all problems. Try to solve as many as you can. Try to bootstrap yourself: your answers to some questions may prove useful in solving other problems. *Write your answers clearly, so that I can read them.* Follow the Honor Code. Good luck.

Problem 1 (10 points). Find all solutions of the following congruence:

$$4x \equiv 8 \pmod{16}$$

Solution (page 145, Exercise 11 c): We want $16 \mid (4x-8)$, so $4 \mid (x-2)$; that is, $x-2 = 4k$ for some k. The values of x (mod 16) are $x = 2, 6, 10, 14$.

Problem 2 (10 points). Find all solutions of the following system of congruences:

$$\begin{cases} x + 5y \equiv 5 \pmod{10}, \\ 5x + 3y \equiv 1 \pmod{10}. \end{cases}$$

Solution (page 146, Exercise 24):

Multiplying the first equation by 2 gives $2x \equiv 0 \pmod{10}$, so $x \equiv 0, 5 \pmod{10}$. Substituting $x \equiv 0$ in the second equation gives $3y \equiv 1 \pmod{10}$, which has the unique solution $y \equiv 7$. Since the pair $x = 0$, $y = 7$ satisfies the first equation, we have one solution in this case. Next, substituting $x \equiv 5$ in the second equation gives $3y \equiv 6 \pmod{10}$, which has the unique solution $y \equiv 2 \pmod{10}$. Since the pair $x = 5$, $y = 2$ satisfies the first equation, we have another solution. In summary, the solutions are:

$$x = 0,\ y = 7 \text{ and } x = 5,\ y = 2.$$

Problem 3 (6 points). True or false:

(a) A ***tournament*** is a digraph in which every two distinct vertices x, y are connected to each other by exactly one arc: xy or yx.

(b) A tournament is ***transitive***, if and only if for any of its three vertices x, y and z, the existence of arcs xy and yz implies that there is an arc xz.

(c) A tournament is transitive if and only if it has a unique Hamiltonian path.

(d) A ***matching*** in a graph is a set of edges that do not share vertices with each other.

(e) Vertices that are incident with matching edges are called ***saturated***.

(f) A matching is ***perfect*** if and only if it saturates all vertices.

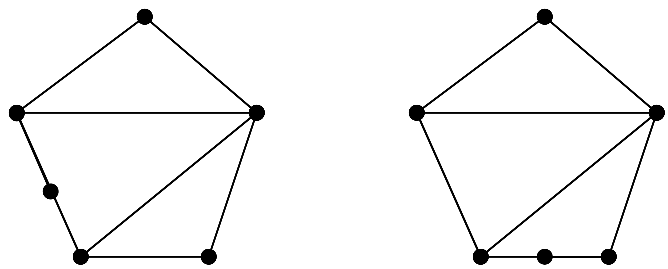
Answer: all true.

Problem 4 (20 points). Establish whether each pair of graphs are

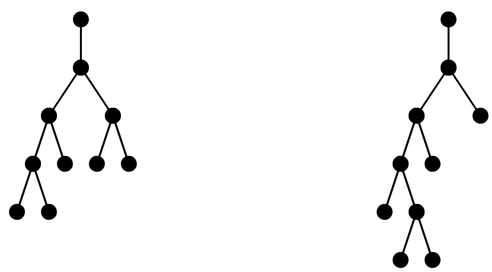
- isomorphic?
- homeomorphic?

If isomorphic, show isomorphism by labeling corresponding vertices with same labels.

(a)



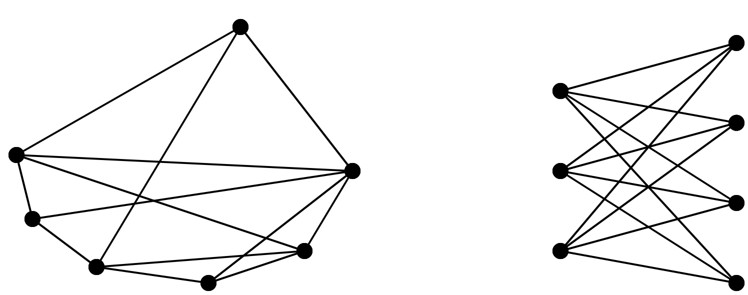
(b)



(c)

(d)

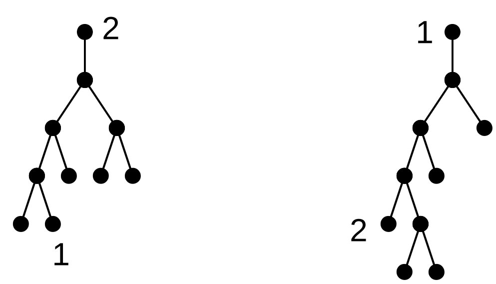
(e)



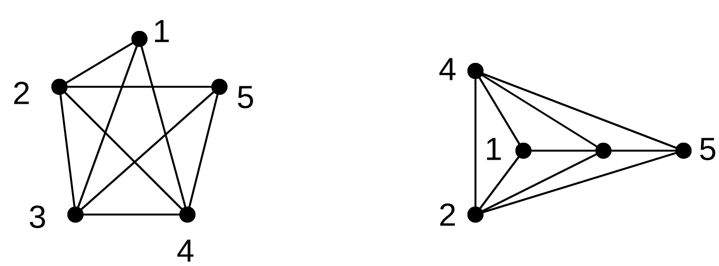
Solutions:

(a) Not isomorphic: the 4-cycle on the left shares two edges with two triangles, which is not the case in the right graph. Therefore, the graphs cannot be superimposed with 1-1 correspondence between all elements. Homeomorphic: deletion of vertices of degree 2 reduces both graphs to the same graph.
(b) Isomorphic (and therefore homeomorphic) as graphs (see figure below).
(c) (page 412, Figure 13.2): Isomorphic (and therefore homeomorphic): see figure below.
(d) Not isomorphic: the vertex of degree 3 is in the middle of the chain on the right, but not on the left. Therefore, the graphs cannot be superimposed with 1-1 correspondence between all elements. Homeomorphic: deletion of vertices of degree 2 reduces both graphs to the same graph.
(e) (page 300, Exercise 5 a): Not homeomorphic, and therefore not isomorphic. The graph on the left, but not on the right, has a vertex of degree 5. This feature cannot be altered by adding and/or deleting vertices of degree 2 on existing edges, since these operations do not alter degrees.

(b)



(c)



Problem 5 (10 points).

(a) Is it possible to represent K_5 as a plane graph? What about $K_{3,3}$? Please, explain why.

(b) Represent K_5 with one deleted edge as a plane graph with all edges drawn as segments of a straight line.

Solutions:

(a) Kuratowski theorem says that a graph is planar if and only if it has no subgaph isomorphic to K_5 or $K_{3,3}$; therefore, both K_5 and $K_{3,3}$ are not planar and therefore they cannot be represented as plane graphs.

(b) See Problem 4, Figure (c) above.

Problem 6 (10 points). George is examining three graphs: G_1, G_2 and G_3. He gives correct arguments that G_1 is not isomorphic to G_2, and G_2 is not isomorphic to G_3.

(a) Can George conclude that G_1 is not isomorphic to G_3?

(b) What would be the answer to (a), if George found G_2 isomorphic to G_3?

Solution (page 303 Exercises 18, 19):

(a) No. The fact that two graphs are not isomorphic to a third one gives no information about their mutual isomorphism: in principle they can be isomorphic. This does not contradict the transitivity of graph isomorphism.

(b) The answer would be "Yes". As an equivalence relation, isomorphism of graphs is symmetric and transitive. Therefore, if G_1 is isomorphic to G_3, and G_2 is isomorphic to $G_{3,}$ then G_1 is isomorphic to G_2, which contradicts the finding of George that G_1 is not isomorphic to G_2. Therefore, G_1 is not isomorphic to G_3.

Problem 7 (15 points).

Recall that a Gray code of length *n* can be viewed as a cycle composed of all 2^n possible strings of zeros and ones of length *n*, such that any two adjacent strings in this cycle differ precisely by one bit. Here is an example of a Gray code of length 2:

00
01
11
10

Recall that Dirac theorem states that if a graph *G* has $n > 2$ vertices, and every vertex has degree $\geq \frac{n}{2}$, then *G* is Hamiltonian.

For what values of *n* does the Dirac theorem allow you to resolve the question of existence of a Gray code of length *n*? Please, explain why.

Solution

A Gray code of length *n* is a Hamiltonian cycle of the graph G_n whose vertices are all $N = 2^n$ strings, and whose edges correspond to mutations of one string into another by a single bit flip. Therefore, the question of existence of the Gray code of length *n* is the question of whether G_n is Hamiltonian. The degree of every vertex in G_n is *n*, and the condition of the theorem is $n \geq \frac{N}{2} = 2^{n-1}$, which is satisfied for $n \leq 2$. Therefore, the theorem may be helpful in proving the existence of the Gray code of length 2 only. Because the theorem has only a one-sided implication ("if", not "iff"), we cannot infer directly from it that a Gray code of length *n* does not exist for any $n > 2$ (in fact, there is an elegant proof by induction that it does exist for any natural $n > 2$).

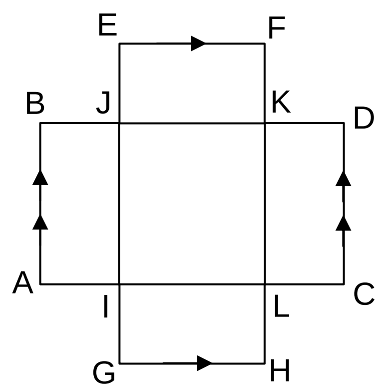
Problem 8 (20 points).

A connected, orientable surface can be created by the following transformation of the polygon ABJEFKDCLHGI shown in the Figure: (i) the edge AB is glued with the edge CD, (ii) the edge EF is glued with the edge GH. The directions of glued arrows in both cases coincide. The rest of edges together form one open boundary of the glued surface.

Using a triangulation of this surface and the Euler formula, compute the Euler characteristic χ and the genus g of the surface.

Hint: the Euler formula is $\chi = v - e + f = 2c - 2g - b$

The glued surface is topologically isomorphic to a torus with a hole in it (an understanding of this fact is not necessary for solving the problem).



Solution

One possible triangulation that is actually shown in the Figure consists of 5 distinct facets: ABJI, JEFK, KDCL, GILH, IJKL. It includes 14 distinct edges: IJ, JK, KL, LI, BJ, JE, FK, KD, CL, LH, GI, IA, AB=CD and EF=GH. There are 8 distinct vertices: A=C, B=D, E=G, F=H, I, J, K, and L. Therefore, in the Euler formula $v = 8$, $f = 5$, $e = 14$. The surface is connected, therefore, $c = 1$. The surface has one continuous open boundary: AIGJBKFL, therefore, $b = 1$. As a result, using the Euler formula, we have:

$$\chi = f - e + v = 5 - 14 + 8 = -1 = 2c - 2g - b = 2 - 2g - 1 = 1 - 2g \quad \Rightarrow \quad g = 1, \chi = -1.$$

Problem 9 (14 points). Prove that a tree with at least one branch has at least two leaves.

Proof: The sum of node degrees in a tree is twice bigger than the number of branches. In a tree of n nodes there are n-1 branches, and therefore the sum of node degrees is 2(n-1). Each leaf has degree 1, and each other node has degree ≥2. Therefore, if a tree of n nodes had no leaves or only one leaf, then its sum of node degrees would be ≥ 2n-1 which is greater than 2(n-1). Therefore, a tree must have at least two leaves.

Alternative proof: From an arbitrarily selected branch of the tree, start growing a path (extending both ends) until the process terminates (it must terminate, because the graph is finite). A termination cannot happen at an internal node, when all available branches lead to previously visited nodes: this would imply that the tree has a cycle, which is impossible by the definition of a tree. Therefore, the process necessarily terminates at two distinct leaves.

Problem 10 (10 points). In a paraffin molecule C_kH_{2k+2} , $k \in \mathbb{N}$, each hydrogen atom H is bonded to 1 carbon atom, and each carbon atom C is bonded to 4 atoms. Explain why the graph of chemical bonds in a paraffin molecule must be a tree.

Solution: According to the theorem, if a connected graph of n vertices has n-1 edges, then this graph is a tree. Suppose that a paraffin molecule C_kH_{2k+2} is represented as a connected graph with vertices corresponding to atoms and edges corresponding to chemical bonds. Then the total number of vertices is $n = k + 2k + 2 = 3k + 2$, and the total number of edges is computed as follows: $2e = 4k + 2k + 2 = 6k + 2 = 2n – 2$, $e = n – 1$. Therefore, the graph is a tree.

Problem 11 (10 points).

Suppose your task is to put a lower bound on the minimal weight of a Hamiltonian cycle in a given fully connected graph G of n = 100,000 vertices with randomly generated, all positive weights W. Would you use the Prim algorithm as a part of your approach? If yes, then how? If no, then what algorithm(s) would you use and why?

Hint: Recall that the Traveling Salesman Problem is known to be NP-complete, while the Prim algorithm allows you to construct a minimum spanning tree in a polynomial time $O(n^2)$.

Solution: In general, finding the minimum Hamiltonian cycle in G defined above is practically impossible, because of the NP-completeness. Therefore, an estimate based on a minimal spanning tree weight found by the Prim algorithm in $O(10^{10})$ steps would be useful, if the minimum spanning tree weight can be related to the minimum Hamiltonian cycle weight. Here is how they can be related. Removal of any edge from a Hamiltonian cycle turns it into a spanning tree for the same graph. Therefore, the weight of a minimum Hamiltonian cycle is greater or equal to the weight of a minimum spanning tree plus the minimal weight of an edge. Moreover, removal of any vertex from the graph G turns G into G' and the minimal Hamiltonian cycle of G into a spanning tree of G'. Therefore, the weight of a minimum Hamiltonian cycle of G is greater or equal to the weight of a minimum spanning tree of G' plus the minimal weight of two edges incident with the deleted vertex. Using any of these cases with the Prim algorithm used to construct a minimum spanning tree would allow to generate a lower bound on the weight of the minimum Hamiltonian cycle. By trying different realizations of these cases, one may improve the estimate.

Problem 12 (10 points). Prove that a Hamiltonian graph has a perfect matching, if and only if the number of vertices in the graph is even.

Proof:

($\rightarrow$) A matching is perfect if and only if it saturates all vertices. Therefore, a perfect matching partitions the set of vertices into pairs. Therefore, a matching cannot be perfect if the number of vertices is odd. Equivalently, a perfect matching implies that the number of vertices is even.

($\leftarrow$) A Hamiltonian graph has a cycle that includes all vertices. Given that the number of vertices is even, it is always possible to match each vertex with its nearest neighbor on the cycle.

Problem 13 (15 points). Prove that a tournament is transitive if and only if it has no cycles.

Proof (Page 360 Exercise 11):

(←) Suppose a tournament is not transitive. Then, without loss of generality, there are three vertices, *x*, *y*, *z*, such that there are two arcs, *xy*, *yz*, but the arc *xz* is missing. In a tournament, every two vertices must be connected by an arc. Therefore, there is an arc *zx*. Therefore, there is a cycle (*x*, *y*, *z*). It follows that the absence of cycles implies the transitivity of the tournament.

(→) Suppose a tournament is transitive and has a cycle. If the cycle includes more than 3 vertices, then, by transitivity, it must have shortcuts. Therefore, there is a shorter cycle for each cycle of length >3. Therefore, because the graph is finite, there must be a cycle of length 3, which means that the tournament is not transitive: a contradiction. Therefore, the tournament has no cycles.

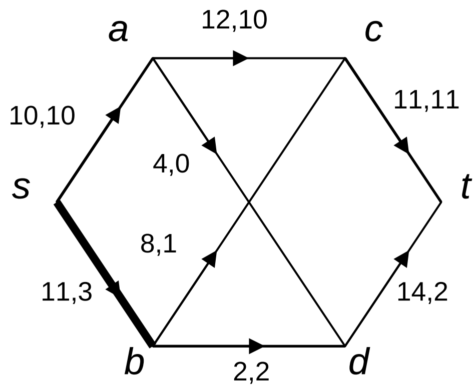
Problem 14 (10 points). Compute the chromatic number χ of the following graphs:

(a) K_5,

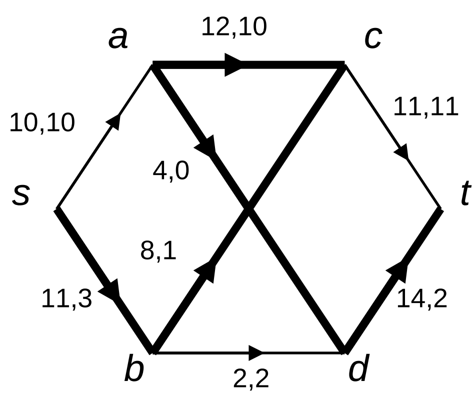
(b) $K_{3,3}$.

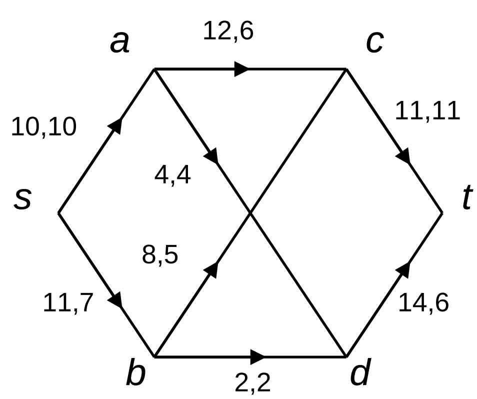
Answer: (a) 5, (b) 2.

Problem 15 (15 points). Find a maximum (*s*,*t*)-flow in the graph shown below.
Hint: eliminate the slack of a flow-augmenting chain.



Solution: The flow-augmenting chain is *sbcadt*. The slack of the chain is 4. Eliminating the slack produces a maximum flow (see the figure below). The flow value is maximal by the max-flow min-cut theorem, because it is equal to the capacity of the cut that goes through the arcs *bd*, *ac* and *ct*.





Problem 16 (15 points). Observe that two "magic numbers", a = 625 and b = 9376, have the following properties:

$$a^2 \equiv a \pmod{n}, \quad b^2 \equiv b \pmod{n}, \quad a + b \equiv 1 \pmod{n}, \quad ab = 0 \pmod{n} \quad \text{with } n = 10^4.$$

These properties allow you to perform additions modulo n, subtractions modulo n and multiplications modulo n in "encrypted format", as follows. Suppose, for example, that the task is to compute

$$R = uv - w + (x - y)\, z \qquad (*)$$

using two independent calculators, given arbitrary integer numbers u, v, w, x, y, z. The first calculator receives every input number $u..z$ after its multiplication modulo n by a, then performs calculations (*) and obtains the result P. In parallel, the second calculator receives every input number $u..z$ after its multiplication modulo n by b, and then uses (*) to compute the result Q. Now, given P and Q, how the two should be combined in order to "decode" R?

Hint: it takes one operation modulo n.

Answer: $R = P + Q \pmod{n}$. This follows immediately from the properties of addition modulo n and multiplication modulo n.

Extra credit problems

Solving these problems is not a requirement and will not add to your score above the maximum of 200 points.

Extra Credit Problem 1 (25 points)

Recall that Catalan numbers obey the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad \text{with} \quad C_0 = 1.$$

Find an analytical expression for the generating function of Catalan numbers:

$$f(x) = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + \ldots$$

Hint: Express f^2 in terms of f.

Solution:

$$f^2(x) = \left(\sum_{n=0}^{\infty} C_n x^n\right)^2 = \sum_{n=0}^{\infty} C_n x^n \sum_{m=0}^{\infty} C_m x^m = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} C_k C_{n-k}$$

$$= \sum_{n=0}^{\infty} x^n C_{n+1} = \frac{1}{x}\sum_{n=0}^{\infty} x^{n+1} C_{n+1} = \frac{1}{x}\sum_{n=1}^{\infty} x^n C_n = \frac{1}{x}\left(-1 + \sum_{n=0}^{\infty} x^n C_n\right) = \frac{f-1}{x};$$

$$xf^2 - f + 1 = 0, \quad C_0 = 1 \quad \to \quad f(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Extra Credit Problem 2 (25 points)

Observe a pattern in the following examples:

$$\begin{aligned}
5^2 &= 2\underline{5}\\
25^2 &= 6\underline{25}\\
625^2 &= 390\underline{625}\\
0625^2 &= 39\underline{0625}\\
90625^2 &= 8212\underline{890625}\\
890625^2 &= 793212\underline{890625}\\
2890625^2 &= 835571\underline{2890625}\\
12890625^2 &= 1661682\underline{12890625}\\
212890625^2 &= 4532241\underline{8212890625}\\
8212890625^2 &= 674515724\underline{18212890625}\\
18212890625^2 &= 331709384\underline{918212890625}\\
918212890625^2 &= 843114912509\underline{918212890625}
\end{aligned}$$

Let $A = \{ a_n \}$ be the sequence of leading digits of the above numbers, i.e., the sequence that starts with $a_1 = 5$, $a_2 = 2$, $a_3 = 6$, … and let A_n be the decimal number $(a_n \ldots a_1)_{10}$. Prove by induction that the above table can be continued indefinitely, i.e., that there is an infinite sequence of decimal digits $A = \{a_1, a_2, \ldots\}$ with $a_1 = 5$, such that

$$(a_n \ldots a_1)_{10}^2 \equiv (a_n \ldots a_1)_{10} \pmod{10^n} \quad \forall n \in \mathrm{N}. \qquad (*)$$

Solution: The induction hypothesis is that for a given natural n, A_n satisfies (*). Now we will prove that the induction hypothesis implies that $A_{n+1} = (a_{n+1}\, a_n \ldots a_1)_{10}$ also satisfies (*), if we take as a_{n+1} the $(n+1)$th digit of $(A_n)^2$. Indeed,

$$\begin{aligned}
&A_n^2 \equiv A_n \pmod{10^n}\\
&A_n^2 \pmod{10^{n+1}} = (a_n \ldots a_1)_{10}^2 \pmod{10^{n+1}} = (a_{n+1} a_n \ldots a_1)_{10} = A_{n+1}\\
&A_{n+1}^2 \pmod{10^{n+1}} = A_n^2 \pmod{10^{n+1}} + 2a_{n+1} A_n \pmod{10^{n+1}}\\
&\qquad\qquad = \left(A_n^2 + 2a_{n+1} a_1 10^n\right) \pmod{10^{n+1}}\\
&\qquad\qquad = A_n^2 \pmod{10^{n+1}} = A_{n+1}
\end{aligned}$$

because $2a_1 = 10$. Therefore, by induction, given that the induction hypothesis is true for $n = 1$, it is also true for any natural n.

Extra Credit Problem 3 (25 points)

A certain king decided to determine who of his two wise men is wiser. To do this, he arranged a contest between them that is to be taken only once. The king announces the rules, according to which he may use up to two black hats and at most one white hat in this contest. The king blindfolds the wise men, then puts a hat (exactly one, either black or white) on the head of each of them, then clamps, signaling that the participants may open their eyes and look at each other. Nobody can or is allowed to see the color of his hat, while everybody can see the color of the opponent's hat. No conversation is allowed, except for naming the color of your hat, and doing this only once. The winner is the one who correctly names the color of his hat first.

Imagine that you participate in this contest. You may assume that your opponent is wise and is determined to win the contest. You may assume that the king is smart and that his intention is indeed to find out who of the two is wiser. You see a white hat on your opponent. *Five minutes have passed*, nobody broke the silence. What color is your hat?

Hint: Try to think outside the box of the proposition calculus.

<u>*Solution:*</u>

The color is white. The short argument is that the king is smart, and his intention is to find out who of the two is wiser in a single contest; therefore, the king must put both participants in equal conditions. The long argument below deals with potential contra-arguments, presenting the solution in the form of a real-time-like line of reasoning of an imaginary participant of the contest.

The rules state that there is only one white hat, and I see it; therefore, my hat must be black. But in this case my opponent would reason as follows.

"I see a black hat. I know that my hat is either white or black. Suppose it is white. Then my opponent would see immediately that the only white hat is not on him, and he would infer that he is wearing a black hat. Then I see no reason why he should not name his color, given that he is a wise man and wants to win. But he is silent. Therefore, my hat must be black."

After this reasoning, which should take less than five minutes for a wise man, I do not see why he should not announce that he wears a black hat, given that he is a wise man and wants to win. But he is silent. What possibilities are left? One possibility is that the king does not follow the rules: he is the king, after all. Besides, using two different colors would be inconsistent with his intention to determine who of us is wiser in a *fair* contest. In any case, if the king is indeed determined to find out who of us is wiser, then he must put us in equal conditions; therefore, he must use *two hats of the same color*. Therefore, my hat is white. But, if my hat is white, and my opponent can see it, then why did not he say immediately that his hat is black? For example, because he could reason as follows.

"The rules say that there is only one white hat, and it is not on me. Therefore, my hat is black. I can say this immediately – or I can wait and say this later. What do I loose if I decide to wait (and my hat is black)? In this scenario, I trick my opponent into believing that I do not see a white hat. This false belief would encourage him to give a wrong answer, making me the winner. Therefore, I can afford to wait and still win. Waiting gives me one advantage: I can think more. Therefore, I decide to wait one minute."

Then, for example, he may come to the following line of reasoning.

"The hypothesis that my hat is black is inconsistent with the intention of the king to determine who of us is wiser: the contest must be fair, and because it is given only once, the king must use two hats of the same color. Given this argument, I should admit a possibility that the king is cheating, and that my hat is white. If my hat is white, and my opponent did not say by now that his hat is black, then he was reasoning similarly to my own reasoning, and by now he is probably waiting for me to give a wrong answer, which means that I probably can wait one minute, no longer. If, however, my hat is black, then very soon (probably in less than a minute) my opponent will conclude that his hat is black and say that – unless he gets suspicious that I am pretending. Therefore, I should probably give him one more minute, before I confidently announce that my hat is white."

Giving five minutes to a wise man to infer from seeing a black hat on me that he is wearing a black hat is more than can be justified in this situation; however, the *fact* that five minutes have passed and he did not break the silence makes me confident in my conclusion. The bottom line of my consideration of the above *example* of his possible line of reasoning is that the hypothesis that the king is cheating (and that my hat is white) is not inconsistent with the observed behavior of my opponent, who may be *pretending* that he is seeing a black hat. The actual reason that allows me to say confidently that I wear a white hat is my belief in the intention and the wisdom of the king.