Computation of the Arc Length

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Overview

A ruler is a common household tool which enables us to measure everyday items with a series of straight line approximations. This article describes how to compute the arc length of a planar curve using a similar set of straight line approximations. Example Matlab code is provided in “arcLengthDemo.m”.

Derivation

Overview

The derivation for this proof originates from [1].

The proof is broken into three steps.

1. Approximate the arc by line segments.
2. Use the mean value theorem to justify converting the approximation into a derivative.
3. Convert the sum of segments into an integral.

The basic idea is to break the arc into a series of linear segments and sum those segments while applying a limiting process to shrink the line segments into a continuous set of points which can be symbolically integrated. The final result will be the formula for arc length for a function f(x) over the interval $[x_1,x_2]$ which is given by:

$$s=\int_{x_1}^{x_2}\sqrt{1+\left(f'\left(x\right)\right)^2}\,dx$$

Assumptions

1. The function is a planar curve. This assumption is needed to simplify the problem so we are integrating over one variable. If the curve was not planar this proof could be expanded to multiple dimensions or a parametric equation could be used.

2. The function is continuously differentiable over the interval of interest. This assumption will be needed when we have to justify our conversion from the sum of segments into an integral.

Proof

Assume we have a planar curve $y_i = f(x_i)$, which is continuously differentiable over the interval [a,b] and we wish to compute the arc length over this interval.

First we break the interval up into a series of disjoint segments
$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

The distance metric for each segment will be the standard Euclidean distance as shown below.

$$d_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$
where
$$\Delta x_i = x_i - x_{i-1}$$
$$\Delta y_i = f(x_i) - f(x_{i-1}) = y_i - y_{i-1}$$

The segments are summed together to approximate the arc length. Only under the limit of n approaching infinity does the sum become the true arc length.

$$s = \text{true arc length}$$
$$s \approx \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$
$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

Now we have to find a way to convert the delta segment lengths into something we can symbolically manipulate. The conversion is provided by the mean value theorem. The Mean Value Theorem will let us relate the function's derivative to the ratio of the y and x segments which is the slope of the segment. This relationship will allow us to symbolically manipulate the equation. Basically the Mean Value Theorem says is that there is a point c_i which lies between $[x_i, x_{i-1}]$ such that the derivative of the function evaluated at c_i equals:

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i}$$

This theorem allows us to replace the slope of the segment with the function's derivative as shown below:

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\left(f'(c_i)\right)^2}\,\Delta x_i$$

We are getting close to the finish line but we need to convert the summation into an integral to complete the proof. The assumption that we use to convert the summation into an integral is that the function is continuous. By being continuous it means that the function can be integrated so in the limiting process the summation can be converted into an integral:

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\left(f'(c_i)\right)^2}\,\Delta x_i \xrightarrow[\textit{assumption}]{\text{continuity}} \int_{x_1}^{x_2} \sqrt{1+\left(f'(x)\right)^2}\,dx$$

Which is what we were trying to prove.

Matlab Implementation

Overview

We now use the results from the previous proof to evaluate the implementation in Matlab and its results when compared to an analytic solution. The two methods we implemented in Matlab are a summation of segments method and a method that computes the arc length integral with Matlab's quad function. The example code is given in "arcLengthDemo.m".

Analytic Solution

Assume we wish to find the arc length of the function which meets the required assumptions. The function is $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$ which is shown in Figure 1.

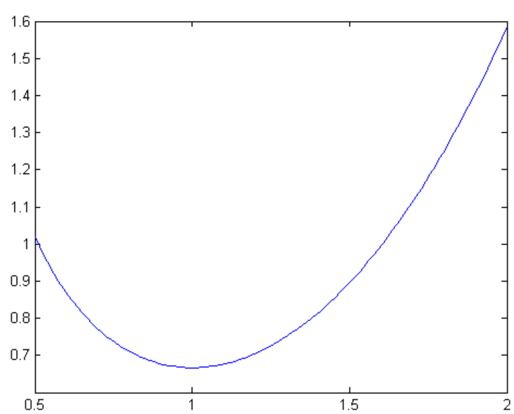

Figure 1: Graph of f(x)

This is solved by taking the derivative and then using the arc length formula.

$$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

The arc length is:

$$s = \int_{\frac{1}{2}}^{2} \sqrt{1 + \left(f'(x)\right)^2}\,dx = \int_{\frac{1}{2}}^{2} \sqrt{1 + \left(\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right)^2}\,dx$$

$$= \int_{\frac{1}{2}}^{2} \sqrt{1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)}\,dx$$

$$= \int_{\frac{1}{2}}^{2} \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\,dx$$

$$= \int_{\frac{1}{2}}^{2} \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)dx$$

and solving the integral we get:

$$s = \frac{1}{2}\left(\frac{x^3}{3} - \frac{1}{x}\right)\Bigg|_{\frac{1}{2}}^{2} = \frac{1}{2}\left(\frac{8}{3} - \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{24} - \frac{2}{1}\right) = \frac{1}{2}\left(\left(\frac{16}{6} - \frac{3}{6}\right) - \left(\frac{1}{24} - \frac{48}{24}\right)\right)$$

$$= \frac{52}{48} + \frac{47}{48} = \frac{99}{48}$$

$$= \frac{33}{16}$$

This value will be considered the truth and used to evaluate the effectiveness of the two Matlab approaches.

Sum Approximation

We create Matlab code which will implement the approximation for arc length shown below:

$$s \approx \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \sqrt{\left(\Delta x_i\right)^2 + \left(\Delta y_i\right)^2}$$

Assume we define two vectors in Matlab, one which contains the function values, y, evaluated at points given by the vector x. Both vectors are column vectors of size N. So we have:

$$\mathbf{y} = \left[\, f(x_1) \quad f(x_2) \quad \cdots \quad f(x_N) \,\right]^T$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^T$$

The differences can be found by taking the deference of the vectors and summing their element wise squares. This is given by:

$$\mathbf{\Delta x} = \left[\ x_2 - x_1 \quad x_3 - x_2 \quad \cdots \quad x_N - x_{N-1}\ \right]$$
$$\mathbf{\Delta y} = \left[\ y_2 - y_1 \quad y_3 - y_2 \quad \cdots \quad y_N - y_{N-1}\ \right]$$

$$s \approx \sum_{i=1}^{n} \sqrt{\left(\Delta x_i\right)^2 + \left(\Delta y_i\right)^2}$$

This is given in Matlab as:

```
dx=diff(x);
dy=diff(y);

a1=sum(sqrt(dx.^2+dy.^2));
```

After running arcLengthDemo.m we see the sum approximation was off by -0.0021378%

Quadrature Integration

We will create Matlab code which will approximation the actual integral.

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

This is done by using the quad function shown below.

```
limitsOfIntegration=[1/2,2];
fd = @(x) 1/2*(x.^2 - 1./ x.^2);
f_arc=@(x) sqrt(1+fd(x).^2);
a2=quad(f_arc,limitsOfIntegration(1),limitsOfIntegration(2));
```

After running arcLengthDemo.m we see the quad integration was off by 4.8576e-006%

References

1. Larson, Hostetler, Edwards, Calculus, Fourth Edition, D. C. Heath and Company.