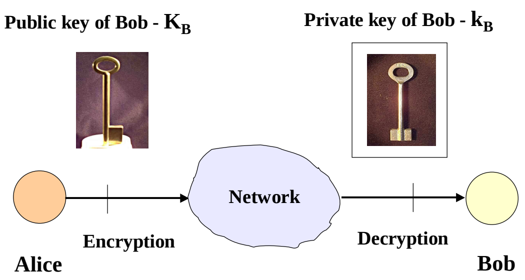
# ECE297:11 Lecture 12

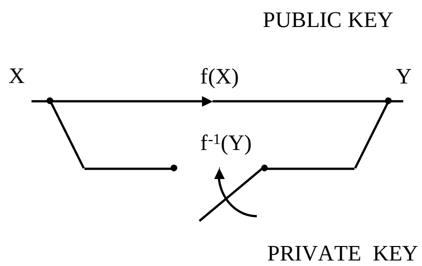
## RSA – Genesis, operation & security

## Public Key (Asymmetric) Cryptosystems




## Trap-door one-way function

Whitfield Diffie and Martin Hellman
"*New directions in cryptography,*" 1976



## Professional (NSA) vs. amateur (academic) approach to designing ciphers

1. Know how to break Russian ciphers
2. Use only well-established proven methods
3. Hire 50,000 mathematicians
4. Cooperate with an industry giant
5. Keep as much as possible secret

1. Know nothing about cryptology
2. Think of revolutionary ideas
3. Go for skiing
4. Publish in "Scientific American"
5. Offer a $100 award for breaking the cipher

## Challenge published in Scientific American

*1977*

**Ciphertext:**

9686 9613 7546 2206 1477 1409 2225 4355
8829 0575 9991 1245 7431 9874 6951 2093
0816 2982 2514 5708 3569 3147 6622 8839
8962 8013 3919 9055 1829 9451 5781 5145

**Public key:**

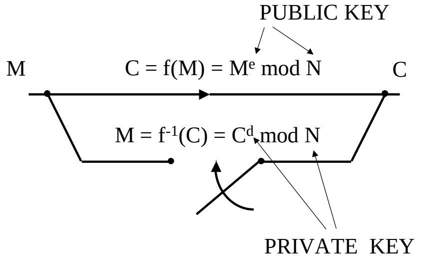
N = 114381625757 888867669235779976146612010218296721242362562561842935706935245733897830597123563958705058989075147599290026879543541

(**129 decimal digits**)

e = 9007

*Award 100 $*

## RSA as a trap-door one-way function




$N = P \cdot Q$ P, Q - large prime numbers

$e \cdot d \equiv 1 \mod ((P-1)(Q-1))$

## RSA keys

**PUBLIC KEY** **PRIVATE KEY**

**{ e, N }** $\leftarrow$ $\not\rightarrow$ **{ d, P, Q }**

$N = P \cdot Q$ P, Q - large prime numbers

$e \cdot d \equiv 1 \mod ((P-1)(Q-1))$

## Why does RSA work? (1)

$$M' = C^d \bmod N = (M^e \bmod N)^d \bmod N \stackrel{?}{=} M$$

decrypted message original message

$$e \cdot d \equiv 1 \mod ((P-1)(Q-1))$$

$$\Updownarrow$$

$$e \cdot d \equiv 1 \mod \varphi(N)$$

Euler's totient function

## Euler's totient (phi) function (1)

**φ(N) - number of integers in the range from 1 to N-1 that are relatively prime with N**

Special cases:

1. P is prime

$$\varphi(P) = P-1$$

Relatively prime with P: 1, 2, 3, …, P-1

2. $N = P \cdot Q$ P, Q are prime

$$\varphi(N) = (P-1) \cdot (Q-1)$$

Relatively prime with N: $\{1, 2, 3, \ldots, P \cdot Q-1\} - \{P, 2P, 3P, \ldots, (Q-1)P\} - \{Q, 2Q, 3Q, \ldots, (P-1)Q\}$

## Euler's totient (phi) function (2)

Special cases:

3. $N = P^2$ P is prime

$\varphi(N) = P \cdot (P-1)$

Relatively prime with N: $\{1, 2, 3, \ldots, P^2-1\} - \{P, 2P, 3P, \ldots, (P-1)P\}$

**In general**

If $N = P_1^{e1} \cdot P_2^{e2} \cdot P_3^{e3} \cdot \ldots \cdot P_t^{et}$

$$\varphi(N) = \prod_{i=1}^{t} P_i^{ei-1} \cdot (P_i-1)$$

## Euler's Theorem

*Leonard Euler, 1707-1783*

$$\forall_{a:\ \gcd(a, N) = 1} \quad \mathbf{a^{\varphi(N)} \equiv 1 \pmod N}$$

## Euler's Theorem - Justification (1)

**For N=10**

$R = \{1, 3, 7, 9\}$

Let a=3

$S = \{ 3\cdot1 \bmod 10,$
$3\cdot3 \bmod 10, 3\cdot7 \bmod 10,$
$3\cdot9 \bmod 10 \}$
$= \{3, 9, 1, 7\}$

**For arbitrary N**

$R = \{x_1, x_2, \ldots, x_{\varphi(N)}\}$

Let us choose arbitrary a, such that $\gcd(a, N) = 1$

$S = \{a\cdot x_1 \bmod N, a\cdot x_2 \bmod N, \ldots,$
$a\cdot x_{\varphi(N)} \bmod N\}$

= rearranged set R

## Euler's Theorem - Justification (2)

**For N=10**

R = S

$x_1 \cdot x_2 \cdot x_3 \cdot x_4 \equiv$
$(a \cdot x_1) \cdot (a \cdot x_2) \cdot (a \cdot x_3) \cdot (a \cdot x_4) \bmod N$

$x_1 \cdot x_2 \cdot x_3 \cdot x_4 \equiv$
$a^4 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4 \bmod N$

$a^4 \equiv 1 \pmod N$

**For arbitrary N**

R = S

$$\prod_{i=1}^{\varphi(N)} x_i \equiv \prod_{i=1}^{\varphi(N)} a \cdot x_i \pmod N$$

$$\prod_{i=1}^{\varphi(N)} x_i \equiv a^{\varphi(N)} \cdot \prod_{i=1}^{\varphi(N)} x_i \pmod N$$

$$a^{\varphi(N)} \equiv 1 \pmod N$$

## Why does RSA work? (2)

$$M' = C^d \bmod N = (M^e \bmod N)^d \bmod N =$$

$$= M^{e \cdot d} \bmod N = \left| \begin{array}{l} e \cdot d \equiv 1 \bmod \varphi(N) \\ e \cdot d = 1 + k \cdot \varphi(N) \end{array} \right| =$$

$$= M^{1+k \cdot \varphi(N)} \bmod N = M \cdot (M^{\varphi(N)})^k \bmod N =$$

$$= M \cdot (M^{\varphi(N)} \bmod N)^k \bmod N =$$

$$= M \cdot 1^k \bmod N = M$$

## Rivest estimation - 1977

The best known algorithm for factoring a 129-digit number requires:

**40 000 trilion years**
**$= 40 \cdot 10^{15}$ years**

assuming the use of a supercomputer being able to perform

1 multiplication of 129 decimal digit numbers in 1 ns

*Rivest's assumption translates to the delay of a single logic gate ≈ 10 ps*

**Estimated age of the universe: 100 bln years = $10^{11}$ years**

## Early records in factoring large numbers

| Years | Number of decimal digits | Number of bits | Required computational power (in MIPS-years) |
|---|---|---|---|
| 1974 | 45 | 149 | 0.001 |
| 1984 | 71 | 235 | 0.1 |
| 1991 | 100 | 332 | 7 |
| 1992 | 110 | 365 | 75 |
| 1993 | 120 | 398 | 830 |

## How to factor for free?

*A. Lenstra & M. Manasse, 1989*

- **Using the spare time of computers, (otherwise unused)**
- **Program and results sent by e-mail (later using WWW)**

## Practical implementations of attacks

**Factorization, RSA**

| Year | Number of bits of N | Number of decimal digits of N | Method | Estimated amount of computations |
|---|---|---|---|---|
| 1994 | 430 | 129 | QS | 5000 MIPS-years |
| 1996 | 433 | 130 | GNFS | 750 MIPS-years |
| 1998 | 467 | 140 | GNFS | 2000 MIPS-years |
| **1999** | **467** | **140** | **GNFS** | **8000 MIPS-years** |

## Breaking RSA-129

**When:** August 1993 - 1 April 1994, **8 months**

**Who:** D. Atkins, M. Graff, A. K. Lenstra, P. Leyland + 600 volunteers from the entire world

**How:** **1600 computers**
from Cray C90, through 16 MHz PC, to fax machines

***Only 0.03% computational power of the Internet***

**Results of cryptanalysis:**

*"The magic words are squeamish ossifrage"*

An award of 100 $ donated to Free Software Foundation

## Elements affecting the progress in factoring large numbers

- computational power
  1977-1993 increase of about 1500 times
- computer networks
  Internet
- **better algorithms**

## Factoring methods

| **General purpose** | **Special purpose** |
|---|---|
| *Time of factoring depends only on the size of N* | *Time of factoring is much shorter if N or factors of N are of the special form* |
| GNFS - General Number Field Sieve | ECM - Elliptic Curve Method |
| QS - Quadratic Sieve | Pollard's p-1 method |
| Continued Fraction Method *(historical)* | Cyclotomic polynomial method |
| | SNFS - Special Number Field Sieve |

## Running time of factoring algorithms

$$L_q[\alpha, c] = \exp((c+o(1))\cdot(\ln q)^{\alpha}\cdot(\ln \ln q)^{1-\alpha})$$

For **$\alpha$=0**

$L_q[0, c] = (\ln q)^{(c+o(1))}$

Algorithm **polynomial** as a function of the number of bits of $q$

For **$\alpha$=1**

$L_q[1, c] = \exp((c+o(1))\cdot(\ln q))$

Algorithm **exponential** as a function of the number of bits of $q$

For **$0 < \alpha < 1$**

Algorithm **subexponential** as a function of the number of bits of $q$

$f(n) = o(1)$ if for any positive constant $c>0$ there exist a constant $n_0>0$, such that $0 \le f(n) < c$, for all $n \ge n_0$

## General purpose factoring methods

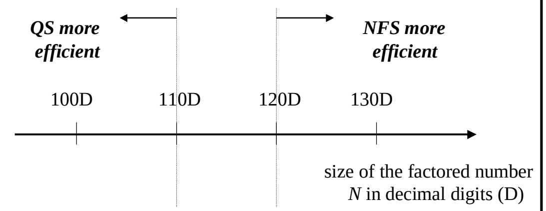
***Expected running time***

**QS**

$L_N[1/2, 1] = exp((1 + o(1))\cdot(\ln N)^{1/2})\cdot(\ln \ln N)^{1/2})$

**NFS**

$L_N[1/3, 1.92] = exp((1.92 + o(1))\cdot(\ln N)^{1/3})\cdot(\ln \ln N)^{2/3})$




## RSA Challenge

RSA-100
RSA-110
RSA-120
RSA-130
RSA-140
RSA-150
RSA-160
RSA-170
RSA-180
.............
RSA-450
RSA-460
RSA-470
RSA-480
RSA-490
RSA-500

Smallest unfactored number

**RSA-150**

Unused awards accumulate at a rate of $1750 / quarter

## Factoring 512-bit number

512 bits = 155 decimal digits
**old standard for key sizes in RSA**

*17 March - 22 August 1999*

Group of Herman te Riele
Centre for Mathematics and Computer Science
(CWI), Amsterdam

First stage 2 months

168 workstations SGI and Sun, 175-400 MHz
120 Pentium PC, 300-450 MHz, 64 MB RAM
4 stations Digital/Compaq, 500 MHz

Second stage

Cray C916 - 10 days, 2.3 GB RAM

## TWINKLE
## "The Weizmann INstitute Key Locating Engine"

*Adi Shamir*, Eurocrypt, May 1999
CHES, August 1999

**Electrooptical device capable to speed-up the first phase of factorization from 100 to 1000 times**

**If ever built it would increase the size of the key that can be broken from 100 to 200 bits**

**Cost of the device (assuming that the prototype was earlier built) - $5000**

## Recommended key sizes for RSA

**Old standard:**

| | |
|---|---|
| **Individual users** | ~~**512 bits** (155 decimal digits)~~ |

**New standard:**

| | |
|---|---|
| **Individual users** | **768 bits** (231 decimal digits) |
| **Organizations (short term)** | **1024 bits** (308 decimal digits) |
| **Organizations (long term)** | **2048 bits** (616 decimal digits) |

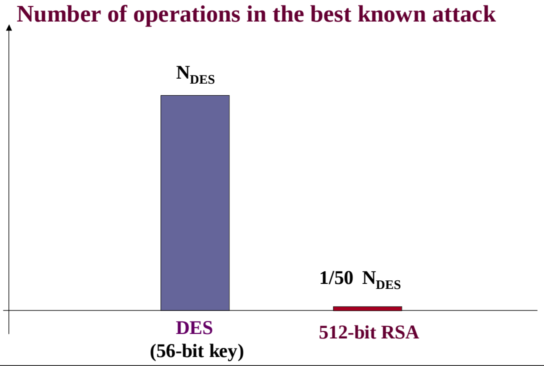
## Keylengths in public key cryptosystems that provide the same level of security as AES and other secret-key ciphers

Arjen K. Lenstra, Eric R. Verheul
*„**Selecting Cryptographic Key Sizes**”*
*Journal of Cryptology*

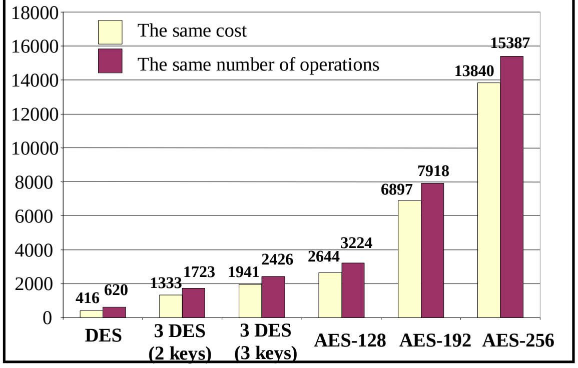
Arjen K. Lenstra
*„Unbelievable Security: **Matching AES Security Using Public Key Systems**”*
ASIACRYPT’ 2001

## RSA vs. DES: Resistance to attack

**Number of operations in the best known attack**




## Keylengths in RSA providing the same level of security as selected secret-key cryptosystems



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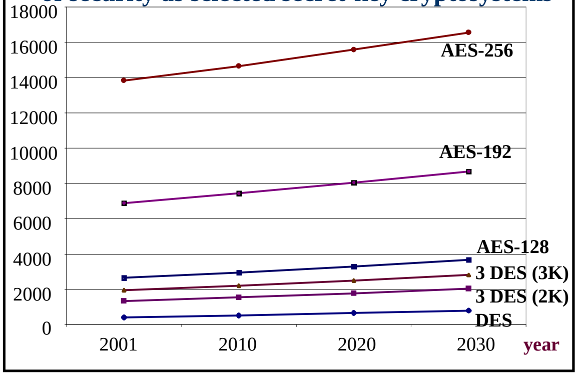



## Practical progress in factorization

**March 2002, Financial Cryptography Conference**

**Nicko van Someren, CTO nCipher Inc.**
announced that his company developed software
capable of breaking 512-bit RSA key
within **6 weeks**
**using computers available in a single office**

## Bernstein's Machine (1)

**Fall 2001**

**Daniel Bernstein**, professor of mathematics
at University of Illinois in Chicago
submits a grant application to NSF
and publishes fragments of this application
as an article on the web

D. Bernstein, ***Circuits for Integer Factorization: A Proposal***

http://cr.yp.to/papers.html#nfscircuit

## Bernstein's Machine (2)

**March 2002**

- Bernstein's article "discovered" during *Financial Cryptography Conference*

- Informal panel devoted to analysis of consequences of the Bernstein's discovery

- Nicko Van Someren (nCipher) estimates that machine costing $ **1 bilion** is able to break **1024-bit RSA** within **several minuts**

## Bernstein's Machine (3)

**March 2002**

- **alarming voices** on e-mailing discussion lists calling for revocation of all currently used 1024-bit keys

- **sensational articles** in newspapers about Bernstein's discovery

## Bernstein's Machine (4)

**April 2002**

**Response of the RSA Security Inc.:**

Error in the estimation presented at the conference;
according to formulas from the Bernstein's article
machine costing
**$ 1 billion** is able to break
**1024-bit RSA** within
**10 billion** x **several minuts** = **tens of years**

According to estimations of Lenstra i Verheul, machine
breaking **1024-bit RSA** within **one day**
would cost **$ 160 billion** in 2002

## Bernstein's Machine (5)

**Carl Pomerance, Bell Labs:**

„…fresh and fascinating idea...”

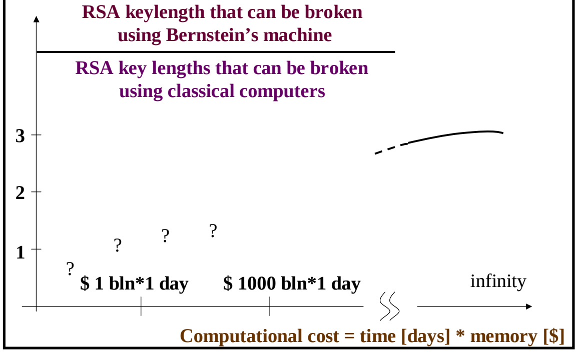
**Arjen Lenstra, Citibank & U. Eindhoven:**

„…I have no idea what is this all fuss about...”

**Bruce Schneier, Counterpane:**

„ ... enormous improvements claimed are more a result of redefining efficiency than anything else...”

## Bernstein's Machine (6)




## RSA Challange

| Lentgh of N in bits | Length of N in decimal digits | Award for factorization |
|---|---|---|
| 576 | 174 | $10,000 |
| 640 | 193 | $20,000 |
| 704 | 212 | $30,000 |
| 768 | 232 | $50,000 |
| 896 | 270 | $75,000 |
| 1024 | 309 | $100,000 |
| 1536 | 463 | $150,000 |
| 2048 | 617 | $200,000 |

**Estimation of RSA Security Inc. regarding the number and memory of PCs necessary to break RSA-1024**

| | |
|---|---|
| **Attack time:** | **1 year** |
| **Single machine:** | **PC, 500 MHz, 170 GB RAM** |
| **Number of machines:** | **342,000,000** |